Simple decisions

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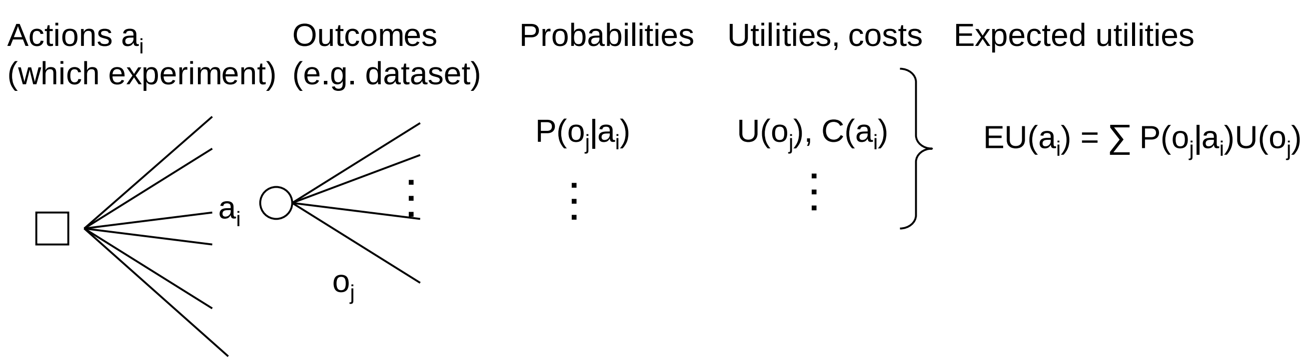
Decision theory
probability theory+utility theory

- Decision situation:
  - Actions a_i
  - Outcomes o_j
  - Probabilities of outcomes $p(o_j \mid a_i)$
  - Utilities/losses of outcomes $U(o_j \mid a_i)$
    - QALY, micromort
  - Maximum Expected Utility Principle (MEU)
    - Best action is the one with maximum expected utility

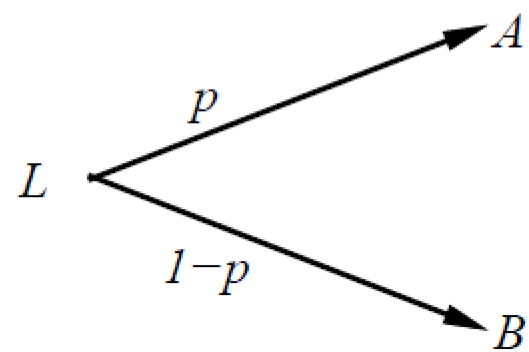
$$EU(a_i) = \sum_j U(o_j \mid a_i)\, p(o_j \mid a_i)$$

$$a^* = \arg\max_i EU(a_i)$$



Preferences

An agent chooses among prizes (A, B, etc.) and lotteries, i.e., situations with uncertain prizes



Lottery $L = [p, A;\ (1-p), B]$

Notation:

$A \succ B$	A preferred to B
$A \sim B$	indifference between A and B
$A \succsim B$	B not preferred to A

Rational preferences

Idea: preferences of a rational agent must obey constraints.

Rational preferences $\Rightarrow$

behavior describable as maximization of expected utility

Constraints:

Orderability

$$(A \succ B) \vee (B \succ A) \vee (A \sim B)$$

Transitivity

$$(A \succ B) \wedge (B \succ C) \Rightarrow (A \succ C)$$

Continuity

$$A \succ B \succ C \Rightarrow \exists p \ [p, A; \ 1-p, C] \sim B$$

Substitutability

$$A \sim B \Rightarrow [p, A; \ 1-p, C] \sim [p, B; 1-p, C]$$

Monotonicity

$$A \succ B \Rightarrow (p \geq q \Leftrightarrow [p, A; \ 1-p, B] \succsim [q, A; \ 1-q, B])$$

An irrational preference

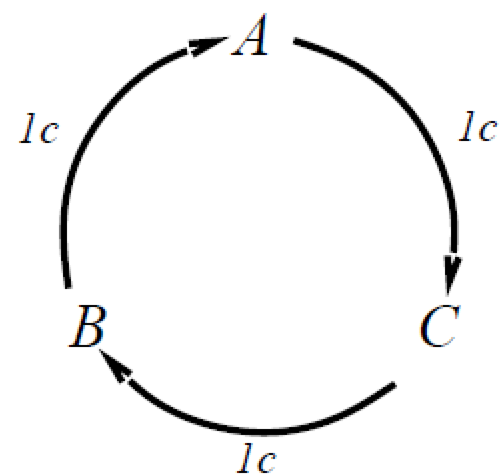
Violating the constraints leads to self-evident irrationality

For example: an agent with intransitive preferences can be induced to give away all its money

If $B \succ C$, then an agent who has C would pay (say) 1 cent to get B

If $A \succ B$, then an agent who has B would pay (say) 1 cent to get A

If $C \succ A$, then an agent who has A would pay (say) 1 cent to get C



Maximizing expected utility

Theorem (Ramsey, 1931; von Neumann and Morgenstern, 1944):
Given preferences satisfying the constraints
there exists a real-valued function U such that

$$U(A) \geq U(B) \quad \Leftrightarrow \quad A \succsim B$$
$$U([p_1, S_1;\ \ldots\ ;\ p_n, S_n]) = \sum_i p_i U(S_i)$$

MEU principle:

Choose the action that maximizes expected utility

Note: an agent can be entirely rational (consistent with MEU) without ever representing or manipulating utilities and probabilities

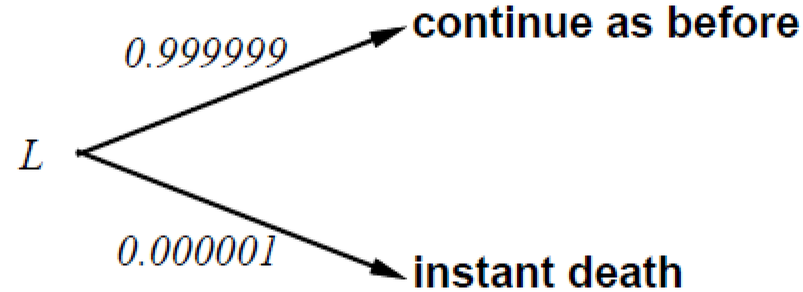
E.g., a lookup table for perfect tictactoe

Utilities

Utilities map states to real numbers. Which numbers?

Standard approach to assessment of human utilities:
- compare a given state A to a standard lottery L_p that has
  - "best possible prize" $u_\top$ with probability p
  - "worst possible catastrophe" $u_\bot$ with probability $(1-p)$
- adjust lottery probability p until $A \sim L_p$

pay $30 ~



Utility scales

Normalized utilities: $u_\top = 1.0$, $u_\perp = 0.0$

Micromorts: one-millionth chance of death
 useful for Russian roulette, paying to reduce product risks, etc.

QALYs: quality-adjusted life years
 useful for medical decisions involving substantial risk

Note: behavior is **invariant** w.r.t. +ve linear transformation

$$U'(x) = k_1 U(x) + k_2 \quad \text{where } k_1 > 0$$

With deterministic prizes only (no lottery choices), only ordinal utility can be determined, i.e., total order on prizes

Disability weights

Haagsma et al.: Assessing disability weights based on the responses of 30,660 people from four European countries, Population Health Metrics (2015) 13:10

Table 3 Estimated disability weights with uncertainty intervals (UI)

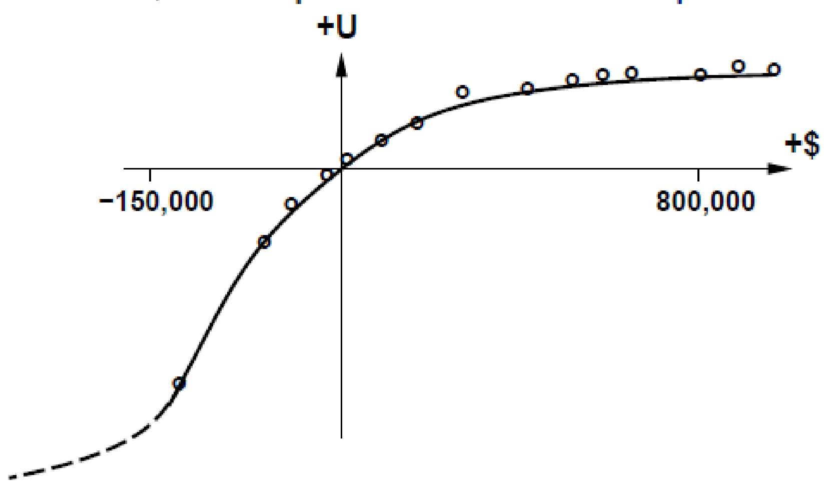
Category[1]		Disability weight (+ UI)		
		Mean	2.5%	97.5%
	Infectious diseases			
Original	Infectious disease, acute episode, mild	0.007	0.005	0.01
Original	Infectious disease, acute episode, moderate	0.051	0.039	0.06
Original	Infectious disease, acute episode, severe	0.125	0.104	0.152
Original	Infectious disease, post-acute consequences (fatigue, emotional lability, insomnia)	0.217	0.179	0.251
Original	Diarrhea, mild	0.073	0.061	0.092
Original	Diarrhea, moderate	0.149	0.12	0.182
Original	Diarrhea, severe	0.239	0.202	0.285
Original	Epididymo-orchitis	0.176	0.143	0.208
Original	HIV cases, symptomatic, pre-AIDS	0.351	0.299	0.394
Original	HIV/AIDS cases, receiving ARV treatment	0.108	0.089	0.132
Original	AIDS cases, not receiving ARV treatment	0.574	0.518	0.635
Original	Ear pain	0.015	0.011	0.019
Original	Tuberculosis, not HIV infected	0.308	0.264	0.353
Original	Tuberculosis, HIV infected	0.383	0.345	0.435
Original	Tuberculosis of vertebrae	0.287	0.245	0.332

Money

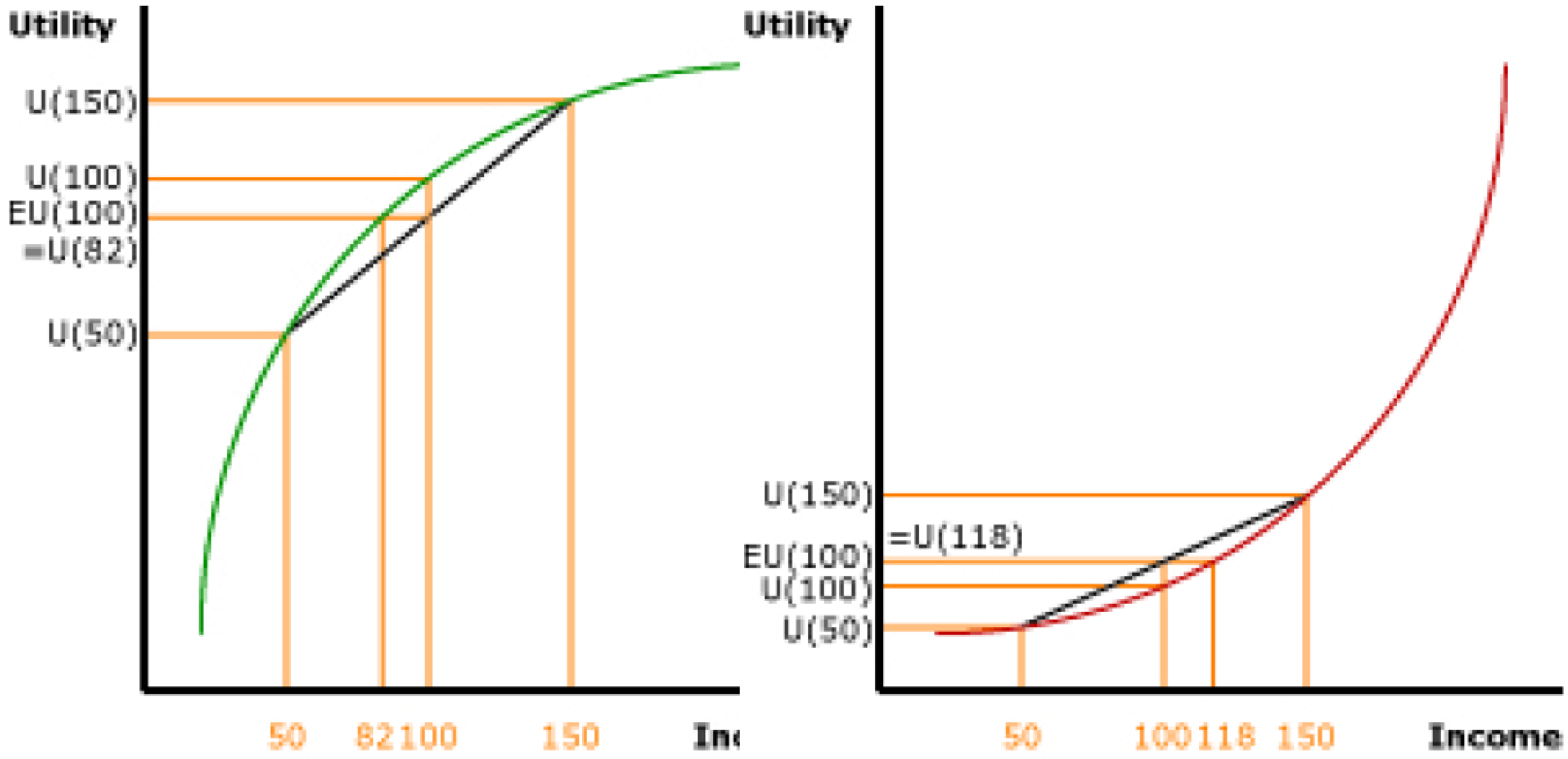
Money does **not** behave as a utility function. Given a lottery L with expected monetary value $EMV(L)$,
usually $U(L) < U(EMV(L))$, i.e., people are risk-averse.

Utility curve: for what probability p am I indifferent between a prize x and a lottery $[p, \$M;\ (1-p), \$0]$ for large M?

Typical empirical data, extrapolated with risk-prone behavior:

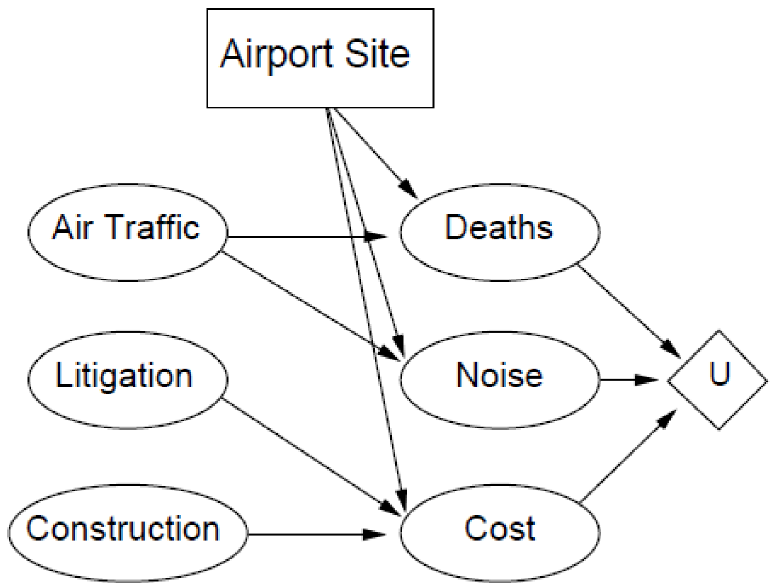


Risk premium in risk aversion and loving



Decision networks (DNs)

Add action nodes and utility nodes to belief networks
to enable rational decision making



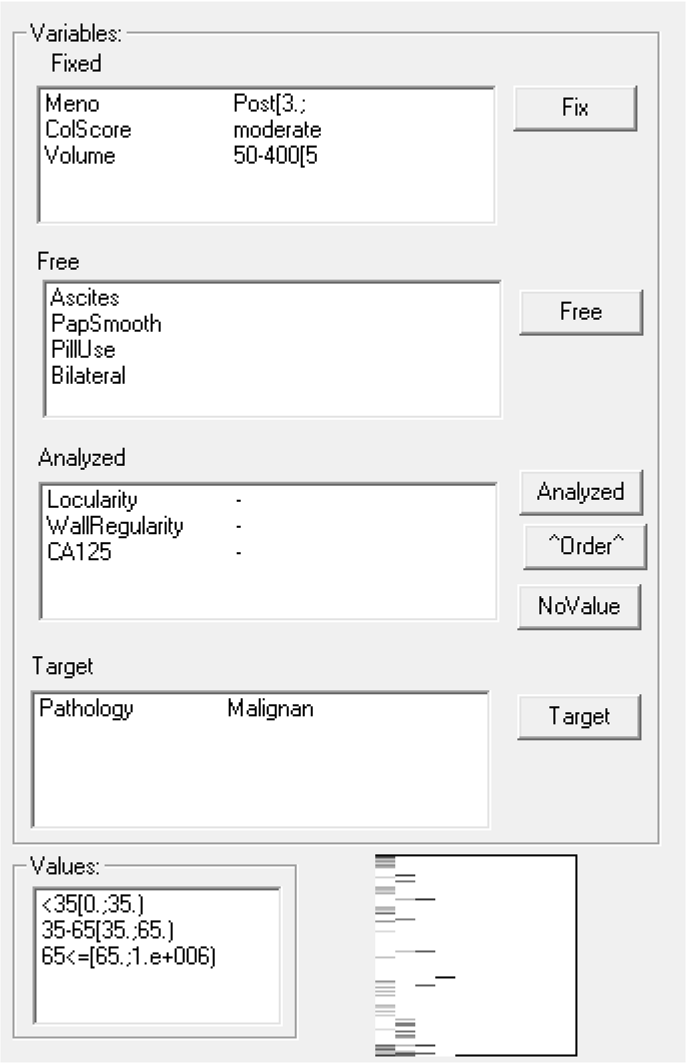
Algorithm:

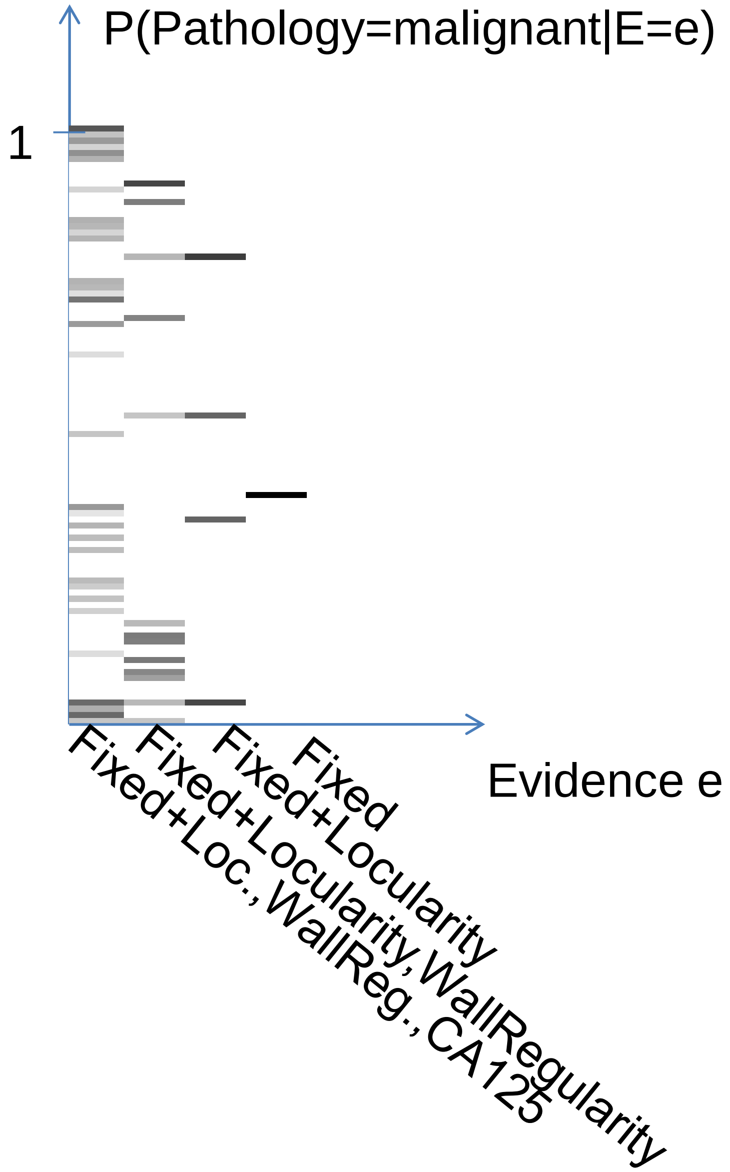
For each value of action node

compute expected value of utility node given action, evidence

Return MEU action

Sensitivity of the inference





Value of information

Idea: compute value of acquiring each possible piece of evidence
Can be done **directly from decision network**

Example: buying oil drilling rights
Two blocks A and B, exactly one has oil, worth k
Prior probabilities 0.5 each, mutually exclusive
Current price of each block is $k/2$
"Consultant" offers accurate survey of A. Fair price?

Solution: compute expected value of information
= expected value of best action given the information
minus expected value of best action without information
Survey may say "oil in A" or "no oil in A", **prob. 0.5 each** (given!)
= [$0.5 \times$ value of "buy A" given "oil in A"
+ $0.5 \times$ value of "buy B" given "no oil in A"] – 0
$= (0.5 \times k/2) + (0.5 \times k/2) - 0 = k/2$

General formula

Current evidence E, current best action α
Possible action outcomes S_i, potential new evidence E_j

$$EU(\alpha|E) = \max_a \Sigma_i\, U(S_i)\, P(S_i|E,a)$$

Suppose we knew $E_j = e_{jk}$, then we would choose $\alpha_{e_{jk}}$ s.t.

$$EU(\alpha_{e_{jk}}|E, E_j = e_{jk}) = \max_a \Sigma_i\, U(S_i)\, P(S_i|E,a,E_j = e_{jk})$$

E_j is a random variable whose value is *currently* unknown
$\Rightarrow$ must compute expected gain over all possible values:

$$VPI_E(E_j) = \left(\Sigma_k\, P(E_j = e_{jk}|E) EU(\alpha_{e_{jk}}|E, E_j = e_{jk})\right) - EU(\alpha|E)$$

(VPI = value of perfect information)

Properties of VPI

Nonnegative—in **expectation**, not **post hoc**

$$\forall j, E \;\; VPI_E(E_j) \geq 0$$

Nonadditive—consider, e.g., obtaining E_j twice

$$VPI_E(E_j, E_k) \neq VPI_E(E_j) + VPI_E(E_k)$$

Order-independent

$$VPI_E(E_j, E_k) = VPI_E(E_j) + VPI_{E,E_j}(E_k) = VPI_E(E_k) + VPI_{E,E_k}(E_j)$$

Note: when more than one piece of evidence can be gathered, maximizing VPI for each to select one is not always optimal

$\Rightarrow$ evidence-gathering becomes a **sequential** decision problem

Extensions

- Bayesian learning
  - Predictive inference
  - Parametric inference
- Value of further information
- Sequential decisions
  - Optimal stopping (secretary problem)
  - Multiarmed bandit problem
  - Markov decision problem
  - ....