III. WEIGHTED LEAST-SQUARES

ERROR: $e(n) \rightarrow \sum w(n)\, e^2(n)$

$w(n) \geq 0$

LET: $\underline{y}^\circ = \underline{\underline{W}}^{1/2}\underline{y}$

MORE WEIGHT ON INFORMATIVE DATA !

$\underline{\underline{\Phi}}^\circ = \underline{\underline{W}}^{1/2}\underline{\underline{\Phi}}$

$\underline{\underline{W}} = \text{diag}\{w(n)\}$

$\underline{\underline{W}}^{1/2} = \text{diag}\{\sqrt{w(n)}\}$

$(\text{diag}\,\underline{\underline{A}})^T = \text{diag}\,\underline{\underline{A}}$

$$V^\circ = \| \underline{\underline{W}}^{1/2}(\underline{y} - \underline{\underline{\Phi}}\,\underline{\theta})\|_2^2$$

$$= \|\underline{y}^\circ - \underline{\underline{\Phi}}^\circ\underline{\theta}\|_2^2$$

$\frac{\partial V^\circ}{\partial \underline{\theta}} \rightarrow$ LIKE STANDARD

$$\boxed{\begin{aligned}\hat{\underline{\theta}} &= (\underline{\underline{\Phi}}^{\circ T}\underline{\underline{\Phi}}^\circ)\,\underline{\underline{\Phi}}^{\circ T}\underline{y}^\circ \\ &= (\underline{\underline{\Phi}}^T\underline{\underline{W}}\,\underline{\underline{\Phi}})^{-1}\underline{\underline{\Phi}}^T\underline{\underline{W}}\,\underline{y}\end{aligned}}$$

IF WEIGHTING: $\sum \frac{e^2(n)}{w(n)}$ $\quad V^\circ = \|\underline{\underline{W}}^{-1/2}(\underline{y} - \underline{\underline{\Phi}}\,\underline{\theta})\|_2^2$

$\underline{\underline{W}} \rightarrow \underline{\underline{W}}^{-1}$

$$\boxed{\hat{\underline{\theta}} = (\underline{\underline{\Phi}}^T\underline{\underline{W}}^{-1}\underline{\underline{\Phi}})^{-1}\underline{\underline{\Phi}}^T\underline{\underline{W}}^{-1}\underline{y}}$$

IV. UNDETERMINED LEAST-SQUARES

(MANY SOLUTIONS !)

$\underline{\underline{\Phi}} = [\quad\quad]$

\# COLUMNS > \# ROWS

MINIMAL NORM SOLUTION:

(CONSTRAINT OPTIMIZATION) $V = \|\underline{\theta}\|_2^2$ UNDER CONDITION

$\underline{y} = \underline{\underline{\Phi}}\,\underline{\theta}$

$$\frac{\partial}{\partial \underline{\theta}}\Big[\overbrace{\|\underline{\theta}\|_2^2 + \underline{\lambda}^T(\underline{y} - \underline{\underline{\Phi}}\,\underline{\theta})}^{\mathcal{L}}\Big]$$

$$\frac{\partial \mathcal{L}}{\partial \underline{\theta}} = 2\underline{\theta} - \underline{\underline{\Phi}}^T\underline{\lambda}$$

$$\frac{\partial \mathcal{L}}{\partial \underline{\lambda}} = \underline{y} - \underline{\underline{\Phi}}\,\underline{\theta}$$

$$\left.\begin{aligned}\hat{\underline{\theta}} &= \tfrac{1}{2}\underline{\underline{\Phi}}^T\underline{\lambda} \\ \underline{y} &= \underline{\underline{\Phi}}\,\underline{\theta}\end{aligned}\right\} \quad \underline{y} = \tfrac{1}{2}\underline{\underline{\Phi}}\,\underline{\underline{\Phi}}^T\underline{\lambda}$$