Voting Systems*

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*This is a chapter prepared for educational usage by undergraduates at the University of Kansas and for eventual inclusion in a mathematics textbook that is being written with Saul Stahl of the University of Kansas Department of Mathematics. Revisions will be ongoing, so comments & corrections are especially welcome. Please contact Paul Johnson <pauljohn@ku.edu>.

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1 Introduction

We rank everything. We rank movies, bands, songs, racing cars, politicians, professors, sports teams, the plays of the day. We want to know "who's number one?" and who isn't. As a local sports writer put it, "Rankings don't mean anything. Coaches continually stress that fact. They're right, of course, but nobody listens. You know why, don't you? People love polls. They absolutely love 'em" (Woodling, December 24, 2004, p. 3C). Some rankings are just for fun, but people at the top sometimes stand to make a lot of money. A study of films released in the late 1970s and 1980s found that, if a film is one of the 5 finalists for the Best Picture Oscar at the Academy Awards, the publicity generated (on average) generates about $5.5 million in additional box office revenue. Winners make, on average, $14.7 million in additional revenue (Nelson et al., 2001). In today's inflated dollars, the figures would no doubt be higher. Actors and directors who are nominated (and who win) should expect to reap rewards as well, since the producers of new films are eager to hire Oscar-winners.

This chapter is about the procedures that are used to decide who wins and who loses. Developing a ranking can be a tricky business. Political scientists have, for centuries, wrestled with the problem of collecting votes and moulding an overall ranking. To the surprise of undergraduate students in both mathematics and political science, mathematical concepts are at the forefront in the political analysis of voting procedures. Mathematical tools are important in two ways. First, mathematical principles are put to use in the scoring process that ranks the alternatives. Second, mathematical principles are used to describe the desired properties of a voting procedure and to measure the strengths and weaknesses of the procedures. Mathematical concepts allow us to translate important, but vague, ideas like "logical" and "fair" into sharp, formally defined expressions that can be applied to voting procedures.

In this chapter, we consider a number of example voting procedures and we discuss their strengths and weaknesses.

2 Mathematical concepts

2.1 Transitivity.

Real numbers are transitive. Every school child knows that transitivity means that

$$If\, x > y\, and\, y > z,\, then\, x > z \tag{1}$$

The symbol $>$ means "greater than" and $\geq$ means "greater than or equal to." Both of these are transitive binary relations. "Binary" means only two numbers are compared against each other, and transitivity means that many number can be linked together by the binary chain. For example, we can work our way through the alphabet. If $a > b$, and $b > c$, and ... $x > y$, then $a > z$.

If a , b, and c represent policy proposals or football teams being evaluated, we expect that voters are able to make binary comparisons, declaring that one alternative is preferred to another, or that they are equally appealing. The parallel between "greater than" and "preferred to" is so strong, in fact, that we use the symbol $\succ$ to represent "preferred to." The subscript i indicates that we are talking about a particular voter, so $x \succ_i y$ means that voter i prefers x to y and $x \succeq_i y$ means that x is as good as y. If i is indifferent, we write $x \approx_i y$.

Using the binary relations $\succ_i$ and $\succeq_i$, it is now possible to write down the idea of transitive preferences. If a person's preferences are transitive, the following holds:

$$If\, x \succ_i y\, and\, y \succ_i z,\, then\, x \succ_i z \tag{2}$$

This can be written more concisely as

$$x \succ_i y \succ_i z \tag{3}$$

According to William Riker, a pioneer in the modern mathematical theory of voting, transitivity is a basic aspect of reasonable human behavior (Riker, 1982). Someone who prefers green beans to carrots, and also prefers carrots to squash, is expected to prefer green beans to squash. One of the truly surprising–even paradoxical–problems that we explore in this chapter is that social preferences, as expressed through voting procedures, may not be transitive. The procedure of majority rule, represented by the letter M, can compare alternatives so that $x \succ_M y$, $y \succ_M z$, and yet x does not defeat z in a majority election. The fact that individuals may be transitive, but social decision procedures are not, has been a driving force in voting research.

2.2 Ordinal versus Cardinal Preferences

If $x \succ_i y$, we know that x is preferred to y, but we don't know by "how much." These are called ordinal preferences because they contain information only about ordering, and not magnitude. Can the magnitude of the difference be measured? The proponents of an alternative model, called cardinal preferences, have developed ingenious mathematical techniques to measure preferences.

The proponents of ordinal preferences argue that all of the information about the tastes of the voters that is worth using is encapsulated in a statement like $x \succ_i y$. Even if we could measure the gap in the desirability between x and y in the eyes of voter i, it would still not solve the problem that the gaps observed by voters i and j would not be comparable. There is no rigorous, mathematically valid way to express the idea that i likes policy x "twice as much" as voter j, even though such an interpersonal comparison is tempting. As a result, the ordinalists argue that no voting procedure should be designed with the intention of trying to measure "how much" more attractive x is than y. Only the ranking should be taken into account.

The approaches based on cardinal preferences offer the possibility of very fine-grained and, at least on the surface, precise social comparisons. If we had a reliable measure of preferences on a cardinal scale, perhaps it would be possible to say that x is .3 units more appealing than y. The most widely used method of deriving these preferences is the Von Neumann-Morgenstern approach which is used in game theory (see chapter X). The theory that justifies the measurement techniques is somewhat abstract and complicated. Even in a laboratory setting, these preferences have proven difficult to measure reliably.

For whatever reasons, almost all voting procedures in use in the world today employ ordinal methods. We ask voters to declare their favorite or to rank the alternatives first, second, and third. As we discuss the weaknesses of various voting methods, readers will often be tempted to wonder if the use of cardinal preference information might lead to an easy fix. We believe that such quick fixes are unlikely to be workable, but they are often very interesting. The fact that one cannot compare preference scores across individual voters should never be forgotten, because many proposals implicitly assume that such comparisons are meaningful.

XX.2 Exercises

1. Consider 3 lunch items, $steak$, $fish$, and $eggplant$. If Joe prefers steak to fish, we write $steak \succ_{Joe} fish$. Joe also prefers $fish$ to $eggplant$. If Joe's preferences are transitive, should we conclude $steak \succ_{Joe} eggplant$ or $eggplant \succ_{Joe} steak$.

2. Suppose Joe tells us $steak \succ_{Joe} fish$ and $steak \succ_{Joe} eggplant$. Can we say whether $steak \succ_{Joe} fish$ or $fish \succ_{Joe} steak$?

3 The Plurality Problem

By far the most commonly used election procedure in the United States is simple plurality rule. Each eligible voter casts one vote and the winner is the candidate that receives the most votes. In terms borrowed from horse racing, it is sometimes called a "first past the post" procedure. The winner need only have more support than the second-placed competitor. This method is used in elections in other countries (e.g., Great Britain) and it is the method of balloting used in the selection of winners of the prestigious Oscar Awards, which are offered by the Academy of Motion Picture Arts and Sciences.

3.1 Two Candidates? No Problem!

If there are only two candidates, then the plurality rule is a good method of choice. It is, in fact, equivalent to majority rule. Suppose the alternatives are x and y and the voters are $N = \{1, 2, 3, ..., n\}$. Voters for whom $x \succ_i y$ will vote for x. In majority rule, an alternative with the support of more than one half of the voters is the winner. Using the notation $|\{i \in N : condition\}|$ to indicate "number of elements in N for which $condition$ is true", majority rule is stated as:

$$\frac{|\{i \in N : x \succ_i y\}|}{n} > \frac{1}{2} \; implies \, x \succ_M y \tag{4}$$

The voters who oppose x, $y \succ_i x$, or are indifferent, $x \approx_i y$, are viewed as the opposition. In contrast, under plurality rule, which we label P, the candidate that is preferred by a greater number of voters wins. Plurality rule is formally defined as

$$|\{i \in N \,:\, x \succ_i y\}| > |\{i \in N \,:\, y \succ_i x\}| \; implies \, x \succ_P y \tag{5}$$

Note this ignores indifferent voters. In contrast, in a majority system, the indifference is treated as opposition. If indifferent voters exist, but they abstain from voting, then plurality and majority again coincide. Only if indifferent voters are given a way to register their indifference, such as casting a "tie" vote, will the two methods diverge.

There is a mathematical proof of the superiority of majority rule known as May's Theorem (May, 1952). Many people have been confused by the fact that May is actually discussing the system that we have defined as plurality rule. In the system that May calls majority rule, votes in favor of two alternatives are collected and the one with the most favorable votes wins (the indifferent voters are not counted as opposition). May's theorem states that when there are two alternatives, majority (what we call plurality) rule is the only decisive procedure that is consistent with these three elementary properties:

- **anonymity**: each person's vote is given the same weight
- **neutrality**: relabeling the alternatives (switching the titles x and y) in voter preferences causes an equivalent relabeling of the outcome (the alternatives are undifferentiated by their labels, sometimes called *undifferentiatedness* of alternatives)
- the electoral system is positively responsive in the following two senses:
  - **strong monotonicity**: if there is a tie and then one voter elevates x in his preferences (that is, changes from $y \succ_i x$ to xI_iy or $x \succ_i y$, or from xI_iy to $x \succ_y y$), then x must be the winner.

– **weak monotonicity**: if x is the winner and one voter elevates x in his preferences, then x must remain the winner.

The formalization and proof of May's theorem is discussed in the exercises.

May's theorem is one of the most encouraging results because it points our attention at a particular method of decision making and it formally spells out the virtuous properties of the procedure.

3.2 Shortcomings of Plurality Rule With More than Two Candidates

Because majority rule has so much appeal with two candidates, there is a natural tendency to try to stretch it to apply in an election with more candidates. The two most important methods through which this has been tried are **pure plurality rule** and the **majority/runoff election system**. In pure plurality, the winner is the one with the most votes, even if it is only 5 or 10 percent. In the majority/runoff system, all of the candidates run against each other and the winner is required to earn more than one half of all votes cast. If no candidate wins a majority, then the top two vote getters are paired off against each other in a runoff election. This system does not typically include "none of the above" as an alternative in either stage, and so there is no method for voters to register indifference (and hence, majority and plurality rule with two candidates imply the same results).

The two stage majority/runoff system is used in some American state and local elections and in French national elections. When the first stage whittles down the list of candidates, it does not necessarily choose the most popular ones. The candidates who have the widest following might "knock each other out," allowing little known candidates to sneak through. The system puts voters in a difficult position of deciding whether they should vote for their favorite, who might be unlikely to win, or for one of the likely winners. The runoff system is also expensive; it requires the government to hold two full elections (ballots aren't free, after all).

Table 1: Plurality election with 4 candidates

		Voters		
Groups	1	2	3	4
Number of Members	20	24	26	30
Ranking				
first	z	y	x	w
second	x	z	y	z
third	y	x	z	x
fourth	w	w	w	y

It is almost too easy to point out the flaws in pure plurality rule. The most obvious weakness of the plurality rule is that the winner need not have widespread support. Suppose there are four candidates, $\{w, x, y, z\}$ running for mayor. The voters can be divided into groups according to their preferences. In Table 1, the preference orderings of the four groups are illustrated. Note that the largest group, group 4, has 30 members and their favorite is w. This example is designed to emphasize the fundamental problem that a minority, 30 percent of the population, is able to force its will on the rest, and all of the other voters think that w is the *worst possible choice*!

It can even happen that a candidate who would lose to each of the others in a head-to-head competition can win in a plurality election. Note that w would lose against x (because x is preferred by groups 1,2,and 3), y, or z.

Holding a runoff between the top two vote getters would help somewhat, since x would win in a head-to-head competition against w. However, the voters who support z would certainly protest. Note that z is **unanimously** preferred to x! How in the world can such a universally unlikeable character as x win an election?

The plurality problem is a "real life" problem with serious consequences. When a candidate wins an election with less than 50% of the vote, there is always concern that the wrong candidate won. Consider, for example, the fact that, in 1998, voters in the state of Minnesota elected former professional wrestler Jesse "The Body" Ventura as their governor. Ventura won with 37% of the vote, outdistancing the Republican candidate Norm Coleman (34%) and the Democrat Hubert Humphrey (28%). There's no way to know for sure what would have happened if either Coleman or Humphrey had faced Ventura in a one-on-one contest, but there's considerable speculation that either would have defeated him.

Even if you don't care much about who is governor of Minnesota, you probably do care who wins the coveted Grammy Awards for the best songs and albums. The winners are selected by a plurality vote with 5 nominees on the ballot. There is often concern that the best performers do not win. Music reporter Robert Hilburn suggested they consider changing their voting procedure, noting, "Looking over previous Grammy contests, it's easy to see where strong albums may have drawn enough votes from each other to let a compromise choice win. In 1985, two of the great albums of the decade-Bruce Springsteen's "Born in the U.S.A." and Prince's "Purple Rain"-went head to head in the best album category, allowing Lionel Richie's far less memorable "Can't Slow Down" to get more votes" (Hilburn, February 28, 2002). Clever voters might try to outsmart the system, voting for their second-ranked alternative in order to stop a weak candidate from winning. Dishonest voting is euphemistically called strategic voting in the literature.

The plurality problem has been known for centuries. It was a focus of concern in French academic circles in the late 1700s. Two extremely interesting characters are the philosopher/scientists Jean-Charles de Borda and the Marquis de Condorcet. Recall that it was a time of revolution, both in America and in France. The philosophy of democracy was becoming well accepted. The Marie Jean Antoine Nicolas Caritat, Marquis de Condorcet (1743-1794), was an extremely influential philosopher of the Enlightenment, studying not only voting (Condorcet, 1785), but also publishing on the rights of women, slavery and free markets. He was highly placed in academic circles, an eager proponent of revolution against the King and, eventually, elected to the legislature (and then later imprisoned by an opposing faction). He is credited with a comment which translates as, "The apparent will of the plurality may in fact be the complete opposite of their true will" (cited in Mackenzie, 2000b). Condorcet proposed a system of voting in which alternatives were paired off for comparison. Traveling in the same circles as Condorcet was Borda (1733-1799), an explorer, soldier, and scholar whose study of physics and mathematics had wide-ranging impact on science. He agreed with Condorcet on the shortcomings of the plurality system and he also proposed a new election procedure based on rank order voting. His manuscript, which was written sometime between 1781 and 1784 (see Saari, 1994), indicates that he had previously presented his results on June 16, 1770 (Borda, 1781). It appears that Borda had the best of it in the eyes of the French because, after the French Revolution (and until Napoleon took over) the Borda procedure was used in French elections. The Borda count fell into disuse after that, only to be revived in the modern era for rankings in American sports. Variants are used in legislative elections in the Pacific Island countries of Kiribati and Nauru (Reilly, 2002).

The debate between Borda and Condorcet has framed research on elections for two centuries. In the next section we consider Borda's method.

XX.3 Exercises

1. In the American criminal justice system, the members of a jury must agree unanimously in order to reach a verdict. If the result is not unanimous when deliberations end, there is a "hung jury" and a mistrial is declared. Explain why this system does not satisfy the principle of strong monotonicity.

2. Suppose there are 50 shares of stock in a company and the stockholders are allowed to vote on company policy. Each stockholder is allowed to cast one vote per share owned. Does this system violate the criterion of neutrality or anonymity?

3. This example has 50 voters and 3 alternatives.

		Voters	
Groups	1	2	3
Number of Members	24	16	10
Ranking			
first	x	y	x
second	y	z	z
third	z	x	y

(a) Calculate the plurality vote totals for the candidates

(b) Does the plurality winner have a majority of the votes?

(c) Confirm that a runoff election will choose the same winner. What changes could you make in the example in order to cause a difference to emerge between the plurality and majority/runoff elections?

4. In 2003, the voters of California were presented with an interesting choice. First, do you want to remove Governor Gray Davis? Second, if Davis is to be replaced, which of the following 150 candidates would you most prefer? California law states that if the number who answer yes to the first question is more than 50% of the votes cast, then the candidate who receives the most votes will become governor. About 54% of the voters wanted to remove Davis, and the plurality winner of the second vote was body builder & movie star Arnold Schwartzenegger. Consider these questions about the 2003 California election.

   (a) Can we be sure that a majority (or even a plurality) prefer Schwartzenegger to Davis as governor?

   (b) The results indicated that 46% of the people voted to keep Davis, but they lost. What percentage of the vote must the replacement receive in order to take office?

Does that seem sensible?

(c) What incentives did voters have to vote strategically (untruthfully) in the first and second parts of the ballot?

5. Kenneth May showed that a system based on majority rule (in which indifferent voters abstain) is equivalent to a system in which the voters are anonymous, the alternatives are undifferentiated, and the procedure is strongly and weakly monotonic. In May's treatment, the voters consider 2 alternatives x and y and then cast votes by selecting a value from the set $\{-1, 0, 1\}$. A vote of 0 to indicates indifference, while 1 indicates a preference for x and -1 indicates a preference for y.

   (a) Create a mathematical definition of majority rule, assuming that the indifferent voters are ignored.

   (b) Use your definition to consider the following: the winner is x and then one voter who favored y decides she is indifferent. If you apply your procedure for majority rule to the new voter preference data, does the result change?

   (c) Suppose there is a tie and one voter changes her mind to prefer x. Apply your procedure to find out who the winner will be.

   (d) On the basis of your calculations in b and c of this question, do you think your definition of majority rule is consistent with the Monotonicity Principle?

4 Rank-order voting: The Borda Count

If you are a fan of college football, you may be aware of the fact that (until 2006, at least) there has been no national tournament to answer the question "which team is the best." Instead, 100 (or so) college football teams are ranked at the end of the season, and eight of the top teams are selected to play in 4 spotlighted games that are known as the Bowl Championship Series. The top two ranked teams play in the national championship game, and after that game, there is another series of votes intended to rank all of the teams from top to bottom.

The formula used by the BCS to decide which teams should play in the national championship game has been a source of frustration and controversy. In 2001, the University of Nebraska was soundly defeated by the University of Colorado in the final regular season game. That loss prevented Nebraska from winning the Northern division of the Big 12 conference and it was not in the Conference championship game (which Colorado won). Nevertheless, the BCS scores had Nebraska ahead of Colorado and the Nebraska Cornhuskers were selected to face the University of Miami in the championship game. Nebraska was soundly defeated, giving grounds to the contention that the BCS had chosen the wrong team. At the end of the 2003 season, the University of Southern California was not selected for the championship game, even though the polls of writers and coaches placed USC in the top two. Decisions like that have led to a chorus of complaints over the years, summarized aptly a student sports writer at the University of Kansas: division 1A college football uses "what may be the most complicated monstrosity on the planet" (Bauer, December 8, 2004, p. 2) to rank the teams.

The people who run the BCS are not oblivious to the criticism of their methods of ranking teams, and every few years, they make repairs in their voting system. They raise or lower the weight put on the votes cast by coaches, sports writers, and, strangely enough, *computers*! So far, at least, they have not strayed far from the same basic system. Individuals (coaches, writers, and computers) are asked to provide rankings of the top 25 teams. The many individual rankings are summed to build an overall ranking. This is a variant of the method that was proposed by Jean-Charles de Borda (de Borda, 1781).

4.1 The Borda Count

The Borda count, which we label BC, is a method of combining the rankings of many individual voters. Suppose there are k alternatives. The voters rank the alternatives and assign integer scores to them, beginning with 0 (for the worst) stepping up in integer units to $k-1$ (for the best). The rankings assigned by the voters are summed and the alternative with the largest Borda count is the winner. The Borda count generates a complete, transitive social ordering of the alternatives.

In mathematical terms, each voter submits a vector of scores that are keyed to the list of candidates. If the candidates are $\{x, y, z\}$ and voter i submits the vector $v_i = (0, 2, 1)$, that means that the voter thinks y is the best, z is second, and x is the worst. The Borda count earned by a candidate j is the sum of rankings assigned by the voters, $\sum_{i=1}^{n} v_{ij}$. The notation $x \succ_{BC} y$ to indicates that x defeats y in the Borda count. Although we do not concern ourselves with ties in this chapter, we should mention that voters are allowed to register indifference by assigning two alternatives the same value. For example, $(0.5, 0.5, 2)$ indicates that x and y are equally desirable (one vote is split between them), and both are less appealing than z.

The Borda count works sensibly in many situations. For example, if all of the voters agree about the ranking for all of the alternatives, it is easy to see that the Borda count gives a meaningful reflection of their rankings. The rankings assigned by the 3 voters are presented in Table 2. This demonstrates that the "sum of ranks" procedure used in the BCS and other poll-style ranking systems is the same as the Borda count. It is, of course, painfully obvious that x should win. It is the most-preferred alternative for all of the voters and it has the highest Borda count.

A system of this sort has been the basis of many ranking systems used in sports and entertainment. Although there are bitter squabbles about the outcomes of these ranking procedures, the commentary in the public media seldom considers the possibility that there is a fundamental problem in the basic idea of the Borda count itself. Very peculiar–even alarming–things can happen with the Borda count.

Table 2: Borda Count with Homogeneous Voters

			Voter Rankings	
	Voter:	1	2	3
Rankings				
first		x	x	x
second		y	y	y
third		z	z	z

			Borda Votes		Borda Count
	Voter:	1	2	3	
Alternatives					
x		2	2	2	6
y		1	1	1	3
z		0	0	0	0

4.2 A Paradox in the Borda Procedure

The problem that we now introduce has sometimes been called the "Inverted-Order Paradox" or the "Winner Turns Loser Paradox". It is a famous example that has been widely investigated (Riker, 1982, p. 82; Fishburn, 1974). There are seven voters, $\{1, 2, 3, 4, 5, 6, 7\}$ and four alternatives, $\{w, x, y, z\}$. In Table 3, the columns represent the Borda voting vectors assigned by the voters.

The winner, the candidate with the largest Borda count, is y. The overall ranking, from top to bottom, is $y \succ_{BC} x \succ_{BC} w \succ_{BC} z$. This is a strength of the Borda count. It generates a transitive ordering. Nevertheless, in the eyes of its critics, the ordering is complete nonsense.

To bring some life into this discussion, suppose that the alternatives w, x, y, and z are bands competing the prestigious Grammy award. After the votes have been collected, an investigative reporter exposes the fact that group z is composed of frauds who lip-sync their performances. (A group later shown to have been lip-syncing, *Millie Vanilli,* actually did win a

Table 3: Borda count with 4 alternatives

			Borda	Votes						Borda Count
	Voters	1	2	3	4	5	6	7		
Alternatives										
w		3	3	0	0	1	1	3		11
x		2	2	3	3	0	0	2		12
y		1	1	2	2	3	3	1		13
z		0	0	1	1	2	2	0		6

Table 4: Borda count after making z ineligible

			Borda	Votes						Borda Count
	Voter	1	2	3	4	5	6	7		
Alternatives										
w		3	3	1	1	2	2	3		15
x		2	2	3	3	1	1	2		14
y		1	1	2	2	3	3	1		13
z		0	0	0	0	0	0	0		0

Grammy award for Best New Artist after their success in 1989.[1]) After z has been exposed, the Grammy managers declare z ineligible. It would seem logical to expect that, if the ordering was $y \succ_{BC} x \succ_{BC} w \succ_{BC} z$, and we rule the last-place candidate ineligible, then the new ordering among the remaining alternatives should simply be $y \succ_{BC} x \succ_{BC} w$. Band z was a loser, anyway. The winner should still be$y$, shouldn't it?

Sadly enough for the proponents of the Borda count, the answer is no. Assuming that y would still win was a terrible mistake caused by a fundamental flaw in the Borda count. We can re-calculate the votes, supposing either that z is completely eliminated from consideration or that it is forced into last place on all of the ballots (the result is the same, either way). Lets begin by simply moving z to the bottom of each voter's ranking. This will assure that z cannot be the winner because it will finish in last place. The alternatives that were ranked below z are shifted up one notch, as indicated in Table 4.

[1]That award was revoked and Millie Vanilli was removed from the official list of award-winning performers.

After moving z to the bottom of each ballot, the Borda Count indicates that w should be the winner. The ranking from top to bottom, is $w \succ_{BC} x \succ_{BC} y \succ_{BC} z$. Compare that with the previous result and a truly astonishing outcome is revealed: *the ordering of the top three alternatives is exactly reversed*. Instead of winning, y now finishes in third place.

The befuddling aspect of this result is that the Borda ranking of w and x is influenced by the placement of the alternative z. If we are completely focused on the comparison of w and x, then z is an **irrelevant alternative**. When the irrelevant alternative affects the outcome, then the decision making procedure is said to violate the principle of **Independence from Irrelevant Alternatives**, which holds that the social ranking of two alternatives should not be influenced by the placement of other alternatives in the ballots of the voters.

A story might persuade readers that a voting procedure ought to respect the principle of **Independence from Irrelevant Alternatives**. Suppose your friend Daffy and his family have received a gift certificate for a new car, either a Toyota or a Honda. When the Daffy family is focused on those two alternatives, the desirability of all other makes should be irrelevant. Suppose you are discussing this in the car dealer's parking lot and Daffy says out loud, "We prefer the Toyota to Honda. We will go in to pick that one." You figure all the work is done, so you go home. Later, Daffy drives up to your house in a shiny new Honda. You say, "what happened? you preferred Toyota!" Daffy says, "While we were in line, my son heard that Cadillacs have great durability. So we changed from Toyota to Honda!" Could the outcome be more ridiculous? It cannot possibly make sense to have the choice between a Honda and a Toyota depend on the road performance of a Cadillac. And yet, that can happen with the Borda count.

In the example illustrated in Tables 3 and 4, the terminology says z an irrelevant alternative, but it is not, in fact, irrelevant when the Borda count is used. That is the sense in which the Borda count violates the principle of Independence from Irrelevant Alternatives. Although commentators on amateur figure skating competitions do not know the terminology, they have certainly puzzled over it many times because skating uses a Borda-style ranking system. In the 1995 World Figure Skating Championships, Michelle Kwan's surprisingly good performance

Table 5: Borda Count with Three Alternatives

			Borda	Voter						Borda Count
	Voter:	1	2	3	4	5	6	7		
Alternatives										
w		2	2	0	0	1	1	2		8
x		1	1	2	2	0	0	1		7
y		0	0	1	1	2	2	0		4

put her in fourth-place, but caused a reversal in the ranking of the second and third ranked skaters who had already finished before Kwan took the ice (Mackenzie, 2000a).

Another variant of this story begins with three alternatives. In Table 5, the three alternatives are w, x and y. The Borda winner is w. Now, suppose that someone who is in charge of the election desperately wants y to win. That someone is so desperate, in fact, that he is willing to sponsor the entry of a new candidate into the race. The new candidate, z, is chosen very carefully. When z is in the race, the preferences of the voters are the same ones we considered in Table 3. After adding in the losing candidate z, the Borda count selects y, just as the election manager wants. This displays the truly insidious nature of the violation of the principle of irrelevant alternatives. Not only is the decision illogical, but now it is prone to electoral manipulation. Without having precise information about that tastes of the voters, it might be difficult to slide in just the right candidate to bring about the desired outcome. One might mistakenly make the outcome even less desirable.

4.3 Another Paradox: The Borda Winner Is A Loser?

There is another closely-related (equally serious) problem with the Borda count. The Borda count can produce a winner that would lose in a head-to-head election against one of the other alternatives. In Table 3, the Borda winner is y, but in a head-to-head contest between y and x, x would win because it is preferred to y by voters $\{1, 2, 3, 4, 7\}$. This result is not necessarily a sign of trouble in the eyes of the advocates of the Borda count. The Borda method, they ar-

Table 6: Condorcet Winner (w) Rejected by Borda Count

			Borda	Votes				Borda Count
	Voter:	1	2	3	4	5		
Alternatives								
w		3	3	1	1	1		9
x		2	2	0	3	3		10
y		1	0	2	0	2		5
z		0	1	3	2	0		6

gue, averages out the extreme opinions, so that the winner is somehow "more representative" of the overall opinions of the voters.

The problem is actually more serious. When he proposed his voting system, Jean-Charles de Borda drew the attention of a fellow French academic, the Marquis de Condorcet. Condorcet pointed out what he thought was a crippling weakness in Borda's method. Consider Table 6, an example in which there are five voters and four alternatives. The Borda winner is x, but x is defeated in a head-to-head contest with w. In fact, w would defeat each of the other alternatives in a head-to-head contest. Alternative w is preferred to x by voters $\{1, 2, 3\}$, and to y by voters $\{1, 2, 4\}$, and to z by voters $\{1, 2, 5\}$. In Condorcet's honor, we call an alternative that can win a pairwise (head-to-head) contest against each of the other alternatives the **Condorcet Winner**. A voting procedure which does not select the Condorcet Winner is said to violate the **Condorcet Criterion**.

4.4 Inside the Guts of the Borda Count

The ranking created by the Borda count is vulnerable to manipulation by the addition or subtraction of candidates. By working on a few examples, one can gain some good working knowledge of what kinds of changes will make a difference. One of the clearest examples is found by comparing an election with two candidates against an election with three candidates. Suppose there are 100 voters and they are divided into two groups. There are 60 voters in group 1 and they prefer x to y. Formally speaking,

$$Group\,1:\; x \succ_i \; y\, for\, i \in \{1, 2, \ldots, 60\}.$$

There are 40 voters on the other side of the issue,

$$Group\,2:\; y \succ_i \; x\, for\, i \in \{61, 62, \ldots 100\}.$$

The Borda count is summarized in Table 7. Since group 1 has the most voters, its favorite is the winner.

Next, suppose that a third alternative is added, and it is very similar to the loser, y. We want to insert this new alternative into the preferences of the voters so that z is always right next to y. Because z is always grouped together with y, it is sometimes called a **clone**. (One standard for voting procedures is that their outcomes should not be changed by the addition of redundant alternatives (clones). This example is intended to show that the Borda count is not "cloneproof"). There are four possible orderings. The first two new orderings are found by placing z about y in the tastes of group 1:

- Group 1a: $x \succ_i y \succ_i z$
- Group 1b: $x \succ_i z \succ_i y$

The last two are obtained by inserting z into the preferences of group 2:

- Group 2a: $z \succ_i y \succ_i x$
- Group 2b: $y \succ_i z \succ_i x$

By inserting z, which is very similar to y in the eyes of the voters, we thus obtain preferences for four groups. The relative sizes of these groups are referred to as n_{1a}, n_{1b}, n_{2a}, and n_{2b} in Table 8.

Suppose that the original groups are split exactly in half, so $n_{1a} = 30$, $n_{1b} = 30$, $n_{2a} = 20$, and $n_{2b} = 20$. With that even division within the two groups, the Borda count is 120, 90, 90 for x, y, and z, respectively. Since y is similar to z in the eyes of the voters, the result seems plausible because y and z are tied in the Borda count. The original winner, the most favored alternative of the majority group, still wins.

We can make some magic by fiddling with the sizes of the subgroups. Keep in mind that there are always 60 voters for whom x is the most attractive alternative and there are always 40 for whom x is the least attractive. The only manipulation that we consider is the subdivision of the original two groups. (If you use a computer spreadsheet, it is pretty easy to consider lots of conjectures.) For most of the examples that you try, the winner will be x, but there are some exceptions. The exceptions are found when there are many voters who have y preferred to z. That is, if you increase the values of n_{1a} and n_{2b} enough, then the Borda winner will change from x to y. If the groups are divided $n_{1a} = 50$, $n_{1b} = 10$, $n_{2a} = 5$, and $n_{2b} = 35$, then the Borda counts for x, y, and z are 120, 125, and 105. Even though x is still the first-ranked alternative for 60 of 100 voters, the alternative y is the Borda winner.

The algebra of the situation is enlightening. Alternative y will defeat x (i.e., have a higher Borda count) if

$$1 * n_{1a} + 1 * n_{2a} + 2 * n_{2b} > 2 * n_{1a} + 2 * n_{1b}$$

which is easily simplified:

$$n_{1a} + 2 * n_{1b} - n_{2a} - 2 * n_{2b} < 0$$

Keeping in mind that $0 \leq n_{1b} \leq 60$ and $n_{1b} = 60 - n_{1a}$ as well as $0 \leq n_{2a} \leq 40$ and $n_{2a} = 40 - n_{2b}$, the Borda count for y will be superior if

$$n_{1a} + n_{2b} \geq 80$$

In the above example, where we set $n_{1a} = 50$ and $n_{2b} = 35$, this inequality was satisfied and so y was the winner.

Table 7: Borda Count with Two Alternatives

	Borda	Votes		Borda Count
Groups	1	2		
Number of Members	60	40		
Alternatives				
x	1	0		60*1=60
y	0	1		40*1=40

Recall that the premise of this example is that the new alternative, z, is similar to (a clone of) y, and z is inserted into the preferences in the spot immediately preceding or following y. In order for the introduction of z to cause a change in the Borda count that benefits y, it is vital that y be more desirable than z in the eyes of many (at least 80) of the voters. One might criticize this example by pointing out that y and z are not truly similar because a vast majority of the voters see y as more desirable than z, and if they were really similar, the split would be more even. This argument, while interesting, does not rehabilitate the Borda count. Rather, it simply refocuses our attention on the problem of irrelevant alternatives. If the overall ranking of x against y depends on the question of whether y is more desirable than z, then one ought to be cautious about the result from a Borda count. If x loses, its supporters must surely feel abused because the position of their favorite against y is not changed in the transition from Table 7 to Table 8. On the other hand, if y is the loser and x is the winner, then the supporters of y are unhappy and feel unjustly treated because their loss was caused by the high placement of z in the rankings of some voters. It just doesn't seem right. It seems this kind of illogic is fundamental about procedures based on the Borda method of voting. And, in case you were wondering, that means the BCS football ranking system will likely produce exasperating results.

Table 8: Add One Alternative to Table 7

	Borda	Votes				Borda Count
Groups	1	2	3	4		
Number of Members	n_{1a}	n_{1b}	n_{2a}	n_{2b}		
Alternatives						
x	2	2	0	0		$2 * n_{1a} + 2 * n_{1b}$
y	1	0	1	2		$1 * n_{1a} + 1 * n_{2a} + 2 * n_{2b}$
z	0	1	2	1		$1 * n_{1b} + 2 * n_{2a} + 1 * n_{2b}$

4.5 Digression on the Use of Cardinal Preferences

The element of the Borda procedure that is most often thought to be the culprit is the simple method in which preferences are registered. The preferences of the voters are registered as integer-valued rankings. If a voter thinks that x is much better (or just a little better) than y, and y is much (or just a little bit better) than z, the ranking will be submitted as 1-2-3 and the magnitude of preference is ignored. A proponent of cardinal preferences would rather have us figure out a way to precisely measure these differences so that the voter could submit a vote like 1-1.8-2.0 to signify the fact that the first one is the best and the other two are far worse. In the advanced literature on social choice theory, there are in fact some highly prestigious authors, such as Nobel Prize Winners John Nash (1950) and John Harsanyi (1955) who have advocated decision-making based on these fine-grained evaluations. John Nash, the game theorist whose life story was the basis of the movie *A Beautiful Mind,* was probably the first to contend that these cardinal scores should be collected and *multiplied* together to create a social ranking. In his honor, Riker calls this the Nash method (see Riker, 1982).

The use of cardinal scores has intuitive appeal. Harsanyi offers a rigorous mathematical argument in favor of this system, contending that it optimizes the welfare of the community in a utilitarian sense. That may be, but, as Riker demonstrates (1982, p. 110-111), the problem of irrelevant alternatives remains. In fact, the problem may be more severe in the Nash method than in the Borda method. In the Borda method, the fact that voters are restricted to casting votes in integer values limits the extent to which the placement of z might influence the comparison of x and y. In a cardinal election, the votes can range across a continuum, seemingly opening up a much larger range of outcomes that are influenced by irrelevant alternatives.

XX.4 Exercises

1. This example has 65 voters and 3 alternatives.

			Voters	
	Groups	1	2	3
Number of	Members	22	12	31
Ranking				
first		x	y	z
second		y	x	y
third		z	z	x

   (a) Which candidate would win a plurality election?

   (b) What would be the results of a majority/runoff election?

   (c) What would the Borda vectors be and what would be the Borda count?

   (d) Is there a Condorcet Winner in this example?

2. This is a set of preferences for a society of 1000 voters with 4 alternatives

			Voters				
	Groups	1	2	3	4	5	6
Number of	Members	200	150	150	300	100	100
Ranking							
first		z	y	x	w	y	x
second		x	z	y	z	x	y
third		y	x	z	x	w	z
fourth		w	w	w	y	z	w

(a) Is there a Condorcet winner?

(b) Which candidate would win according to the Borda Count?

(c) If you delete candidate w, does the Borda Count change? Does the Condorcet winner change?

3. Calculate the Borda count for the following table. Then add an alternative that finishes in last place in the Borda Count, but also causes the ranking of the top-placed alternatives to change.

			Borda	Votes						Borda Count
	Voter:	1	2	3	4	5	6	7		
Alternatives										
w		2	2	0	0	1	1	2		8
x		1	1	2	2	0	0	1		7
y		0	0	1	1	2	2	0		4

4. Condorcet's original example that was used to criticize the Borda count involved an election with candidates named $Jacques$, $Pierre$, and $Paul$.

			Borda	Votes			
	Groups:	1	2	3	4	5	6
Number	of Voters:	30	1	10	29	10	1
Alternatives							
Pierre		2	2	1	1	0	0
Paul		1	0	0	2	2	1
Jacques		0	1	2	0	1	2

(a) Find the Condorcet Winner

(b) Find the Borda Winner

(c) How many ballots would you have to change in order to make the outcome of the two procedures match up? (Hint: you can solve this by trial and error, but you should not have to. Recall that the Borda count equals the aggregated pairwise vote.)

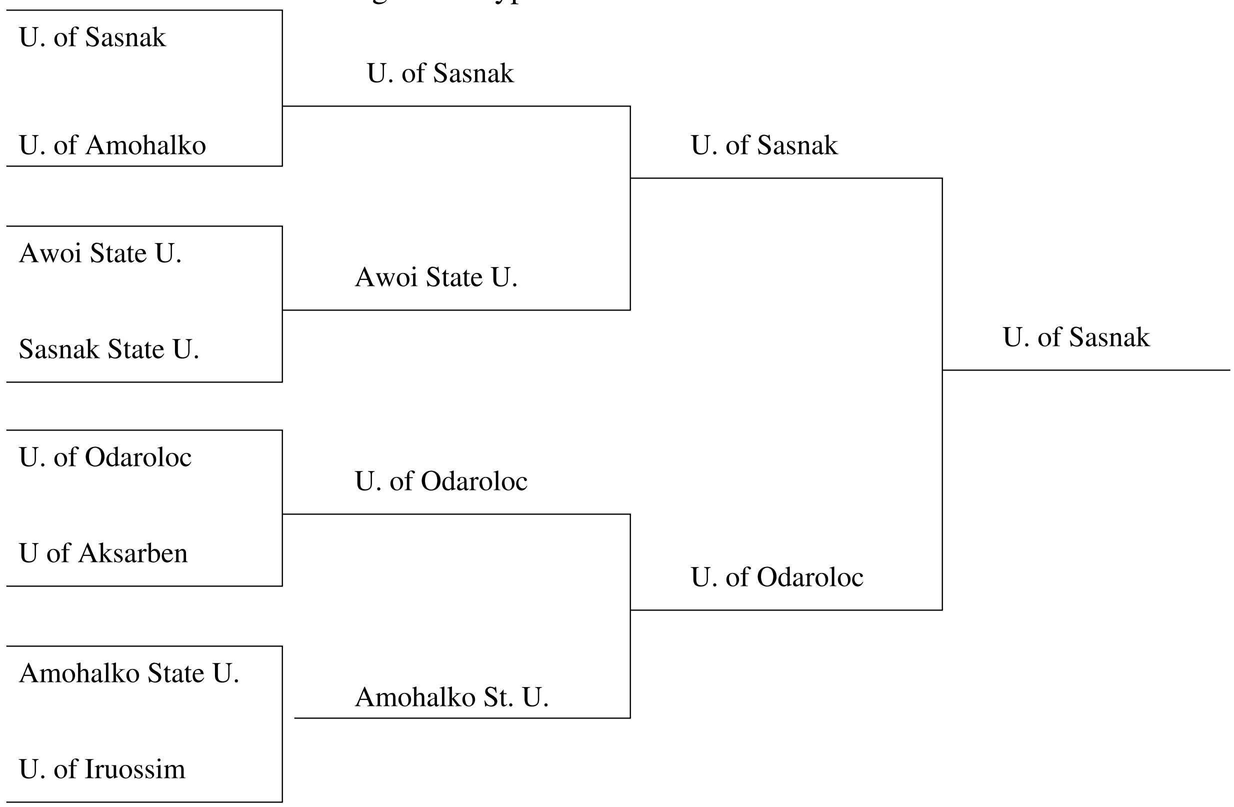
5 Sequential Pairwise Comparisons

The critics of the Bowl Championship Series often contend that college football should adopt a tournament format to select a national champion. A tournament is used in many other college sports as well as professional baseball, basketball and football. Would a tournament solve the controversy over "who's number one?" There are good reasons to be skeptical. A tournament might increase advertising revenue, but we doubt it would put an end to questions about whether the best team was actually ranked number one at the end.

5.1 A Single Elimination Tournament

The teams are eliminated in a series of games, the final of which would be the national championship game. A hypothetical bracket for such a tournament is presented in Figure 1. The

Figure 1: Hypothetical Football Tournament



teams that are defeated at each stage are eliminated, and the winners advance to play each other.

The people who want a Division IA college football tournament usually claim that this is the best way to select the "true champion." In the hypothetical example in Figure 1, the University of Sasnak wins. Whether or not Sasnak is greeted as the "true champion" may be open to question, however. Suppose, for example, that Sasnak could be defeated by Sasnak State, Aksarben, Amohalko State, or Iruossim. Those other teams are not given their "fair chance" to prove their superiority because the tournament structure governs their opportunities. If even one team can defeat the eventual winner of the tournament, it seems we are going to be forever debating what would have happened if the contests had been re-organized. The tournament does not so much solve the problem of deciding "which team is best" as it does create a new set of arguments for people in bar rooms and sports talk shows.

The tournament concept, it turns out, is widely used, not just for sporting events, but for political decision making as well. For example, in American elections, the candidates of the top two political parties face each other in the general election (the "championship game"). Those candidates are the winners of earlier contests. Like the winners of sporting tournaments, these winners are no less open to after-the-fact challenges from contenders who did not get their fair chance.

There is one case in which a tournament structure produces an unequivocal, certain winner. If there is a Condorcet winner, one candidate (or team) that defeats each of the others in a one-on-one competition, then that winner will be the only survivor of the tournament.

5.2 Dominated Winner Paradox

Like the Borda count, the majority rule tournament format can produce some truly bizarre outcomes. The tournament never rejects a Condorcet winner, if there is one, but interesting things can still happen. Perhaps the most well known is the "dominated winner paradox." It is possible that the winner of a tournament can be *unanimously defeated* by another alternative. We mean to say not just that the tournament picks the second-best or third-best, but rather, that the tournament winner is unanimously considered inferior. The preferences in Table 9 are used to illustrate this paradox. There three voters and four alternatives, w,x,y, and z. In the tournament structure that is drawn in Figure 2, there is a a sequence of three votes. First, alternative x is defeated by w. Then, in the second stage, the winner, w is defeated by y, and then the final vote pits$y$ against z. Note that the first outcome to be eliminated, x, is preferred to w by every single voter. [2]

5.3 The Intransitivity of Majority Rule

In the first section of this chapter, we drew the reader's attention to the concept of transitivity. Now that concept comes to the forefront again. There are majority rule examples in which

[2] This is called the "dominated winner paradox" because, in the field of cooperative game theory, one alternative is said to dominate another if every participant prefers it to the other.

Table 9: Dominated Winner Paradox

				Voter	Preferences
	Rankings	Voters:	1	2	3
	best		w	y	x
	second		x	w	z
	third		z	x	y
	fourth		y	z	w
	Sequence of Votes		$w \succ_M x$	$y \succ_M w$	$z \succ_M y$

Figure 2: Tournament Structure for the Dominated Winner Paradox

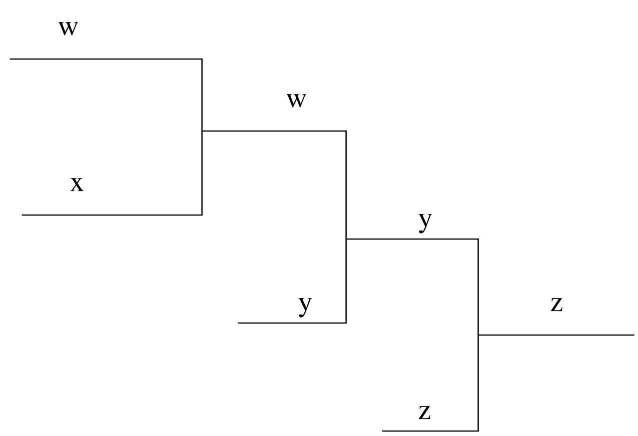


Table 10: Preferences for Three Candidates

		Voter	Groups	
		1	2	3
Favorite Candidate		*Thing 1*	*Cat in the Hat*	*Thing 2*
Second Choice		*Thing 2*	*Thing 1*	Cat in *the Hat*
Third Choice		*Cat in the Hat*	*Thing 2*	*Thing 1*

$x \succ_M y$, and $y \succ_M z$, but (and here's nontransitive part) $z \succ_M x$. Social comparisons sometimes lack the fundamental properties of reason and logic summarized by the principle of transitivity that is present in the tastes of voters.

Consider an example with three candidates (two Democrats, one Republican) and three groups of voters. The Democrats, named *Thing 1* and *Thing 2*, are pitted in a tightly contested electoral campaign, and the winner will face the Republican candidate, named *Cat in the Hat*. Lets suppose there are three equally sized groups of voters. The transitive orderings of the voters are presented in Table 10. Readers will note that this kind of table is slightly different than the one used for the Borda count, but the information that it contains is the same.

It is especially important to note that the preferences of the voters are **transitive**. Consider the preferences in Table 10. For Group 1 we are told that *Thing 1* is preferred to *Thing 2*, and *Thing 2* is preferred to *Cat in the Hat*. Then transitivity of preference implies we are safe in concluding that *Thing 1* is preferred to the *Cat in the Hat* by members of Group 1.

The tournament begins with *Thing 1* and *Thing 2* facing each other and all three groups of voters are allowed to have their say. Since *Thing 1* is preferred to *Thing 2* by groups 1 and 2, *Thing 1* is the winner of that stage. The final stage pits *Thing 1* against *Cat in the Hat*, and with the backing of groups 2 and 3, *Cat in the Hat* wins.

If majority rule were transitive, then we could conclude that *Cat in the Hat* is socially preferred to *Thing 2*. It is a startling and truly paradoxical fact that majority rule does not support that prediction. Majority rule does not obey the property of transitivity.

Imagine what would happen if, in the time-honored tradition of politicians, *Thing 2* hires a large team of lawyers who appeal the case to the highest court. The court might order another election pitting *Thing 2* against *Cat in the Hat*. The *Cat in the Hat* should do everything he can to avoid another vote. Note that groups 1 and 3 prefer *Thing 2* to *Cat in the Hat*, so *Thing 2* would win that final electoral competition.

Even though the individual voter preferences are transitive, the majority rule ordering is not! This peculiar finding, sometimes called **The Voter's Paradox**, means that where we expect to find logic and reason: $x \succ_M y$,$y \succ_M z$, and $x \succ_M z$, instead we find nonsense and irrationality: $x \succ_M y$,$y \succ_M z$, and $z \succ_M x$. We could keep voting forever, with each winner being defeated in turn. This is called a **voting cycle**. A person who is intransitive lacks the fundamental properties of reason and logic and would be incapable of making decisions and managing personal affairs. Should a society that is intransitive be viewed in the same harsh light? For centuries, political scientists have been locked in debate over the issue.

Despite these problems, majority rule is very widely used in legislatures. There are some arguments that can be advanced in favor of the majority rule method and to justify its continued use. By far the most important justification for the use of majority rule is offered by May's theorem, which we have already discussed.

The second reason why majority rule is still widely used, even though cycles are possible, is that cycles are thought to be rare. If there are only three voters with three alternatives, and we ignore the possibility that voters may be indifferent, then we can make a list of all possible "societies." There are 216 possible 3 voter societies, and a cycle can arise in 12, or approximately 5.6% (Johnson, 1998, p. 23). Many computer simulation studies have been done to try to find out how likely a cycle is to arise when the numbers of voters and alternatives are increased. A review of several studies led Riker (1982) to contend that the probability of cycles rises dramatically (near 1.0) as the number of alternatives and voters increase, while Jones et al. (1995) contended that if voter indifference is taken into account, then the probability of cycles is not quite so high.

If cycles are possible in the real world, why don't we observe them more often? Quite simply, we do not use voting procedures that are designed to discover them. Election procedures are generally written so that candidates are eliminated and are not given a second chance. Voting in legislatures is tightly controlled by party leaders who refuse to let the members propose frivolous alternatives in a "fishing expedition". In his discussion of the American Constitutional Convention of 1787, Riker tells the story of a cycle that was revealed when the delegates were attempting to formalize the system for presidential elections (1986, p. 46-7). The cycle caused confusion and uncertainty, and eventually the matter was delegated it to a subcommittee that was trusted to write up something good. That's how the Electoral College was created.

XX.5 Exercises

1. Consider the preferences in problem 1 of section 4. Design a single elimination tournament and find the winner. Is there any way to re-design your tournament so that a different alternative will win?

2. In the following table, we provide preferences for 7 out of 11 voters.

   (a) Fill in the preferences for the last 3 voters in such a way that there is a pairwise voting cycle in which $x \succ_P y$, $y \succ_P z$, and $z \succ_P x$. You can insert any transitive rankings for groups 4, 5, and 6.

			Voters				
	Groups	1	2	3	4	5	6
Number of	Members	3	3	1	1	1	1
Ranking							
first		z	y	x			
second		x	z	w			
third		y	w	z			
fourth		w	x	y			

(b) Choose your favorite letter from the set $\{x, y, z\}$. Design a single-elimination tournament in which your selected alternative wins.

6 Condorcet Methods: The Round Robin Tournament

On the basis of the preceding analysis, the reader should believe that the following claims are correct.

1. If there is a Condorcet Winner, then a single-elimination tournament format will select that alternative.
2. If there is no Condorcet Winner, the tournament winner is determined by the pairings of the alternatives.

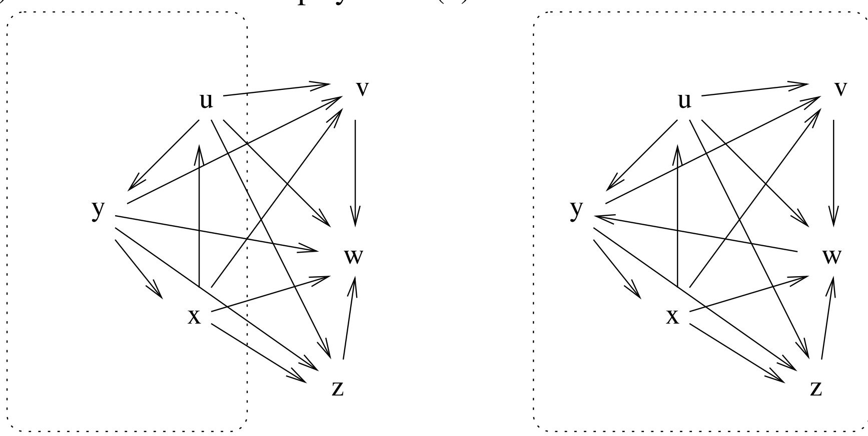
The presiding officer of a town council might exercise the power to set the agenda to advantage some community groups over others. There's a charming essay about an economist and a lawyer who studied social choice theory and then used it to hornswaggle the voters in a club that purchased airplanes for recreational use (Riker, 1986).

To address this problem, Condorcet's approach was to search for a method of voting that will give a meaningful result when all of the pairwise comparisons are considered. Condorcet's suggested that we collect enough information from voters so that we can hold (what is now called) a "round robin" tournament. In a round robin, each alternative faces each of the others in a head-to-head competition.

The **Condorcet Criterion** states that if there is a Condorcet winner–one alternative can defeat each of the others head-on–then it should win. If there is no Condorcet winner, then problem is to find a way to summarize the pairwise information and choose or shape the results into a ranking. Duncan Black, a pioneer of modern social choice research, suggested "The reasons may not seem so overwhelmingly convincing, but we are moving in a region where all considerations are tenuous and fine-spun; and the claims of the Condorcet criterion

Figure 3: The Smith Set

(a) The Smith Set is a "top cycle" (b) The Smith Set includes all alternatives



to rightness seem to us much stronger than those of any other"(Black, 1958). In the time since Condorcet, many different schemes have been proposed to summarize the outcomes of pairwise comparisons. Black suggested the use of a Borda Count. We consider just a few of the many interesting proposals.

6.1 Searching for an Unbeatable Set of Alternatives

Here's an obvious starting point: eliminate undesirable alternatives from consideration. Suppose the alternatives under consideration are $\{u, v, w, x, y, z\}$. In Figure 3, the relationships among the alternatives are represented by arrows. In this type of graph, an arrow from x to y means that x defeats y is a pairwise competition. In panel (a) of Figure 3, note that u, y, and x, the ones that are inside the dotted line, have a special quality. Each of them can defeat each of the others that are outside the dotted line in a head-to-head race. It seems clear that the winner must not be drawn from $\{v, w, z\}$. Each of these can be defeated by each of the proposals in $\{u, x, y\}$. The **Smith Criterion** is based on the idea that, if we can spot some "rejects" in this way, then we ought to make sure they don't win.

Formally, the **Smith set** is defined as the smaller of two sets:

1. The set of all alternatives, X.

2. A subset $A \subset X$ such that each member of A can defeat every member of X that is

not in A, which we call B=$X - A$. Formally, the Smith Criterion states if $x \in A$ and $y \in B = X - A$, $x \succ_P y$.

Note we are using the plurality rule notation here, $x \succ_P y$ (see 6). Indifferent voters are ignored. As we have seen in Figure 3a, the Smith set may include a voting cycle, and that's why some authors refer to it as the "top cycle set." The top cycle includes only the "best of the best." The rest of the alternatives should be rejected.

One major shortcoming of the Smith Criterion is illustrated in Figure 3b. We have turned around a few arrows, so there is no subset of alternatives in which each can defeat the rest. In such a case, the Smith set includes all alternatives. As a result, the Smith Criterion sheds no light at all on the problem. While the Smith criterion can be useful in ruling out alternatives in some examples, but not too many.

6.2 The Win-Loss Record

A second simple approach is to choose a winner on the basis of the win-loss records. This is often called the Copeland rule. We simply subtract the number of losses from the number of wins for each alternative and then rank the alternatives accordingly. If one reads the sports page during professional baseball or football seasons, the win-loss record is a very familiar concept. While simple in principle, this procedure has many shortcomings. Most importantly, if there is a voting cycle, then there are likely to be many ties in the win-loss column. Each alternative will have one win and one loss in the pairwise comparisons, so none can be distinguished from the others. Another problem with this approach its sensitivity to the introduction of clones. One could pad the number of wins for an alternative by making copies of the alternatives that it can defeat. (For Three Stooges fans, if Curly is a loser, then nominate Moe and Larry as well so that the alternatives that defeat Curly also defeat Moe and Larry.)

6.3 Aggregated Pairwise Voting: The Borda Count Strikes Back!

The aggregated pairwise vote is a method of evaluating a tournament. The alternatives accumulate votes in their head-to-head contests and the one with the most votes wins. It creates a transitive overall ranking, even if there is a pairwise majority rule cycle.

Suppose the voters are $N = \{1, 2, \ldots, n\}$ and that none of them are indifferent between any of the alternatives. Recall that the number of voters who prefer x to y can be represented as $|\{i \in N : x \succ_i y\}|$. The aggregated vote for alternative x is the sum of the support it receives against y and z in pairwise contests:

$$|\{i \in N : x \succ_i y\}| + |\{i \in N : x \succ_i z\}|.$$

The aggregated pairwise vote has a number of interesting properties. Readers should easily convince themselves that the following claims are true:

1. A Condorcet winner (one who is undefeated by each of the others) is never ranked last by the aggregated pairwise vote, and

2. A Condorcet loser (one that is unable to defeat any of the others) is never ranked first by the aggregated pairwise vote.

The aggregated pairwise vote is never afflicted by the "dominated winner paradox" that was discussed above. While the Condorcet winner does not always come out on top in an aggregated pairwise vote, that candidate does not suffer the indignity of a last place finish either.

It turns out that these two properties of the aggregated pairwise vote have a powerful implication: the aggregated pairwise vote places a very meaningful limitation on the kinds of paradoxes and peculiarities that can happen. With two alternatives, the aggregated pairwise system is really just plurality rule, so we might write either $x \succ_{AP} y$ or $x \succ_P y$. If the pairwise result is $x \succ_{AP} y$, then a supporter of Condorcet's view of the problem would expect that the social ranking should be $x \succ_{AP} y \succ_{AP} z$ or $z \succ_{AP} x \succ_{AP} y$. As long as the procedure applied to all 3 alternatives has x preferred to y, then everything makes sense. On the other hand, there is trouble if we observe social rankings like $y \succ_{AP} x \succ_{AP} z$ or $z \succ_{AP} y \succ_{AP} x$.

Mathematician Donald Saari offers a way to catalogue these so-called anomalies. For a given set of voters (a preference profile), any voting procedure can be applied to rank the pairs, $\{x, y\}$, $\{y, z\}$, $\{x, z\}$, and also give an ordering of all three alternatives, $\{x, y, z\}$. Since the pairwise comparison is really just a plurality vote, let's use P as the subscript for pairwise comparisons, and we will use AP for the comparisons that take into account 3 or more alternatives. Saari uses the term *word* to refer to a combined set of pairwise and aggregated pairwise rankings, such as

$$word\,1.\, x \succ_P y,\, y \succ_P z,\, x \succ_P z,\, x \succ_{AP} y \succ_{AP} z$$

or

$$word\,2.\, x \succ_P y,\, y \succeq_P z,\, x \succ_P z\,, z \succ_{AP} y \succ_{AP} x$$

There are 351 words possible with 3 alternatives (hint: $3 \cdot 3 \cdot 3 \cdot 13 = 351$). In word 1, there is no anomaly because each of the pairwise plurality decisions is the same as the ordering implied by the aggregated pairwise votes. On the other hand, word 2 is very anomalous. The pairwise comparisons are at odds with the overall ranking. In fact, recalling the first claim above, readers should notice that word 2 is impossible. The aggregated pairwise vote could not assign last place to the Condorcet winner, x.

Saari observes that only 135 out of the 351 words are actually possible with the aggregated pairwise vote applied to three alternatives (1994, p. 186). The driving force behind that result is the fact that the Condorcet winner can't finish last in the aggregated pairwise vote. The complete explanation of this result is given in the focus box 41.

In case you were wondering why the subtitle of this section is "the Borda count strikes back!" we are ready to give the answer: The aggregated pairwise voting procedure is equivalent to the Borda count (Saari, 1994). The strengths of the aggregated pairwise vote are thus inherited by the Borda count. Saari advocates the use of the Borda count for a number of reasons, but the fact that it “is the ’natural’ extension of the standard vote between two candidates” (Saari, 1994, p. 178) to an election with more candidates is one of the most persuasive reasons.

How do we know that the two procedures equivalent? Consider a voter i for whom $x \succ_i y \succ_i z$. The aggregation shows that i casts 2 votes for x (i prefers x to both y and z). Similarly, i votes for y against z, and the aggregation records 1 vote for y. This is, of course, exactly the same as casting a Borda ballot vector $(2, 1, 0)$. The conclusion is that the Borda count has the same information that is collected by the aggregation of pairwise votes.

The fundamental difference between plurality/majority voting and the Borda count is laid bare by this discovery. Recall the definition of plurality rule, which is the same as majority rule if none of the voters are indifferent.

$$Plurality\, rule:\ x \succ_P y\, if\, and\, only\, if\, |\{i \in N : x \succ_i y\}| > |\{i \in N : y \succ_i x\}| \tag{6}$$

Note that the first term on each side of the inequality is the same in the following:

$$\begin{gathered} Borda\, rule:\ x \succ_{BC} y\, if\, and\, only\, if \\ |\{i \in N : x \succ_i y\}| + |\{i \in N : x \succ_i z\}| > |\{i \in N : y \succ_i x\}| + |\{i \in N : y \succ_i z\}| \end{gathered} \tag{7}$$

The Borda comparison between x and y always includes some information about the irrelevant alternative z. The violation of the independence from irrelevant alternatives that we encounter in examples involving the Borda count appears to be inherent within it. At the same time, however, the Borda count avoids both intransitivities and the danger of allowing a Condorcet loser to win an election.

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Feature Box: In *The Geometry of Voting*, mathematics professor Donald Saari argues that the Borda count is less prone to peculiarities than other voting methods that are based on ranked lists. The Borda count admits only 135 out of 351 possible words (combinations of pairwise and listwise outcomes). In contrast, other methods of tabulation allow anything (literally!) to happen. A central part of his argument is the fact that the Borda count is equivalent to aggregated pairwise voting, and in the latter we have observed that a Condorcet winner never places last.

To establish that fact, begin by considering all of the Borda orderings with 3 alternatives. They are divided into three columns, representing social orderings in which the number of ties is zero, one, or two.

1. $x \succ_{BC} y \succ_{BC} z$		7. $x \succ_{BC} y \approx_{BC} z$		13.$x \approx_{BC} y \approx_{BC} z$
2. $x \succ_{BC} z \succ_{BC} y$		8. $x \approx_{BC} y \succ_{BC} z$		
3. $y \succ_{BC} x \succ_{BC} z$		9. $y \approx_{BC} z \succ_{BC} x$		
4. $y \succ_{BC} z \succ_{BC} x$		10. $y \succ_{BC} z \approx_{BC} x$		
5. $z \succ_{BC} x \succ_{BC} y$		11. $z \succ_{BC} x \approx\ _{BC} y$		
6. $z \succ_{BC} y \succ_{BC} x$		12. $z \approx_{BC} x \succ_{BC} y$		

We now have to figure out which pairwise outcomes are consistent with each of these Borda outcomes. The problem is attacked by the timeless method of "divide and conquer."

Consider outcomes 1-6, the ones in which there are no ties. Keeping in mind the fact that a Condorcet winner cannot finish in last place in the Borda count (and the Condorcet loser cannot finish first), we can figure out which pairwise comparisons are possible. We focus on this one case $x \succ_{BC} y \succ_{BC} z$ (and will generalize to the others).

Possible if $x \succ_{BC} y \succ_{BC} z$		Not possible if $x \succ_{BC} y \succ_{BC} z$
$x \succ_P y, y \succ_P z, x \succ_P z$	$x \approx_P y, y \succ_P z, x \succ_P z$	$x \approx_P y, y \succ_P z, x \approx_P z$
$x \succ_P y, y \succ_P z, z \succ_P x$	$x \approx_P y, y \approx_P z, x \succ_P z$	$x \approx_P y, y \succ_P z, z \succ_P x$
$x \succ_P y, y \succ_P z, z \approx_P x$	$x \approx_P y, z \succ_P y, x \succ_P z$	$x \approx_P y, y \approx_P z, z \succ_P x$
$x \succ_P y, z \succ_P y, z \approx_P x$	$y \succ_P x, y \succ_P z, x \succ_P z$	$x \approx_P y, y \approx_P z, z \approx_P x$
$x \succ_P y, y \approx_P z, z \succ_P x$	$y \succ_P x, y \approx_P z, x \succ_P z$	$x \approx_P y, z \succ_P y, z \succ_P x$
$x \succ_P y, y \approx_P z, z \succ_P x$	$y \succ_P x, z \succ_P y, x \succ_P z$	$x \approx_P y, z \succ_P y, z \approx_P x$
$x \succ_P y, y \approx_P z, z \approx_P x$	$y \succ_P x, z \succ_P y, z \succ_P x$	$y \succ_P x, y \succ_P z, z \succ_P x$
$x \succ_P y, z \succ_P y, x \succ_P z$	$y \succ_P x, z \succ_P y, z \approx_P x$	$y \succ_P x, z \succ_P y, z \succ_P x$
$x \succ_P y, z \succ_P y, z \succ_P x$		$y \succ_P x, z \approx_P y, z \succ_P x$
		$y \succ_P x, y \succ_P z, x \approx_P z$

There are 17 sets of pairwise contests that are consistent with $x \succ_{BC} y \succ_{BC} z$. Since the setup of the problem is completely symmetric, the same must be true of Borda orderings 2-6 in **??**. Since 6*17=102, we have found 102 of the 135 legal words.

Next consider the six Borda outcomes that have only one tie, which are items 7-12 in **??**. There are 30 sets of pairwise outcomes that are consistent with these Borda results (five for each one). Considering Borda ordering 7, we find that the following are the legal words.

$$
\begin{aligned}
&1.\quad x \succ_P y,\ x \succ_P z,\ y \approx_M z,\ x \succ_{BC} y \approx_{BC} z\\
&2.\quad x \succ_P y, x \succ_P z, y \succ_P z,,\ x \succ_{BC} y \approx_{BC} z\\
&3.\quad x \succ_P y, z \succ_P x, y \succ_P z, , x \succ_{BC} y \approx_{BC} z\\
&4.\quad x \succ_P y, x \approx_M z, y \succ_P z, x \succ_{BC} y \approx_{BC} z\\
&5.\quad x \approx_M y, x \succ_P z, y \succ_P z, x \succ_{BC} y \approx_{BC} z
\end{aligned}
\tag{8}
$$

With 5 more legal words for each of the 6 Borda outcomes, we have thus added 5*6 = 30 valid words. The proof of this result is left as an exercise for the reader (but it is very informative, and so a complete answer is included at the end of the book).

Finally, consider Borda ordering 13. If the Borda count yields total social indifference,

$x \approx_{BC} y \approx_{BC} z$, only three sets of pairwise matchings are possible:

$$\begin{array}{ll} 1 & x \succ_P y,\ y \succ_P z,\ z \succ_P x \\ 2 & y \succ_P x,\ z \succ_P y,\ x \succ_P z \\ 3 & x \approx_P y,\ y \approx_P z,\ z \approx_P x \end{array} \tag{9}$$

The first two are the cyclical pairwise comparisons, and the final is generalized indifference. With those 3 legal words, our total is now 135.

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6.4 The Schulze Method

If you search the Internet, this one is often found under the unseemly titles "Cloneproof Schwartz Sequential Dropping" or the "Beatpath Method". We have named it for its originator, Markus Schulze, who has been an active proponent of the method since 1997 and has shown that it has many desirable qualities (Schulze, 2003). This is a method that evolved in a sequence of email and Internet postings by a group of enthusiasts who sought to develop workable voting methods that can actually be put to use in "real life." Schulze notes that this method is used in organizations that have an aggregate membership of more than 1,700 and that it is the most widely-used of all of the Condorcet round-robin methods. People who use the Linux operating system might be interested to know that the Schulze method is used in decision making in the Debian Linux project, one of the most widely disseminated Linux distributions.

Of all of the methods we have considered so far, this is the most difficult to understand and interpret at face value. In order to justify it, the proponents do not rely on intuition. Rather, they rely on the mathematical fact that this procedure meets many of the criteria against which voting procedures are judged. If there is a Condorcet winner, this method selects it as the winner. The Schulze method satisfies many of the other most important priorities, including undifferentiatedness (same as neutrality), anonymity, monotonicity, independence of clones (cloneproofness) and several others.

Schulze's method also satisfies a property known as Smith-IIA. Recall the Smith Criterion, which held that the winner should be a member of the Smith Set. Smith-IIA adds the additional stipulation that if a candidate is added in an election, it can only affect the election outcome if it is in the Smith set. If it is not in the Smith set, then it is an irrelevant alternative (so far as the choice among alternatives in the Smith set is concerned) and it should not matter.

The voters submit rankings of as many candidates as they wish, and the ones they do not rank are assumed to be interchangeable and less desirable than the ones they do rank. There are two mathematically equivalent descriptions of the Schulze method for picking the winner in a competitive election. These methods employ the idea of "transitive defeat". A "transitive defeat" (also known as a chain) allows x to assert itself against y through a chain of comparisons, like $x \succ_P w$ and $w \succ_p y$. Even if y defeats x in a head-to-head comparison, if a chain from x to y exists, then x can "transitively defeat" y. You might think of the chain as a sort of "self defense" or "political countermeasure" for electoral candidates. The chains can be quite long, and to save space we might as well write $x \succ_P w \succ_P z \succ_P y$ to mean that x transitively defeats y with w and z in the middle of the chain.

As the old saying goes, a chain is only as strong as its weakest link. The strength of a chain from x to y is measured by the smallest margin of victory between any two alternatives in the chain. The symbol $P[x, y]$ represents the strength of the strongest chain from x to y. If there are two chains leading from x to y, we measure the strength of each one (i.e, find the smallest margin of victory), and $P[x, y]$ is the largest value. If there is no chain from x to y, then the strength is $P[x, y] = 0$.

The "beatpath method" is the one that Schulze uses as the definition of the social ranking process. The aim is to find an alternative with the strongest chains through which it can defeat all other alternatives. If the chain from x to y is stronger than the chain from y to x, $P[x, y] > P[y, x]$, then x disqualifies y from further consideration. If we collect up all the alternatives that are not disqualified, then we have a set of "potential winners." If we are lucky enough to have only one candidate left, then it is the beatpath winner. If we have several left, then a tie breaking procedure is needed. Quite frankly, the emphasis in this method is to narrow the alternatives down to this set of unbeaten candidates, and the tie breaking procedure is something of an afterthought.

To bring this down to earth, consider the application of this method to choosing a national football champion after gathering votes from sports writers and coaches. The voters rank three undefeated teams, Auburn (AU), the University of Southern California (USC), and the University of Oklahoma (OU). In 4, the arrow from USC to OU indicates that USC is preferred to OU by 44 of the voters. Here, we are using the number of voters as the indicator of the strength of the defeat (we could use the margin instead). Note there is a cycle, where OU is ranked higher than AU (strength=22) and AU is preferred to USC (strength=18). This example is particularly simple because there is only one path from each alternative to each of the others. The beatpaths are:

1 $P[OU, AU] = 22$

2 $P[OU, USC] = 18$

3 $P[AU, USC] = 18$

4 $P[AU, OU] = 18$

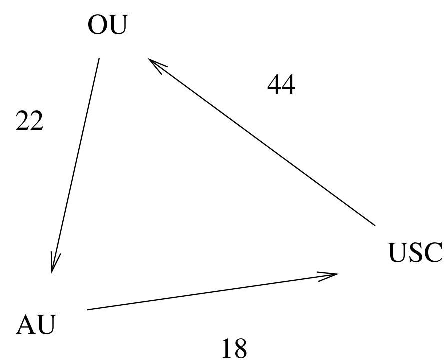
5 $P[USC, OU] = 44$

6 $P[USC, AU] = 22$

Because $P[USC, OU] > P[OU, USC]$and $P[USC, AU] > P[AU, USC]$, USC is the beatpath winner.

The implications of the beatpath calculations are difficult to grasp for many people. Why does the procedure satisfy so many of the desirable qualities? How can we make sense of it? Shortly after Schulze proposed his method in 1997, it was shown that it is equivalent to a method in which we sequentially measure the strength of the candidates and then remove the weak ones, and then re-evaluate. The sequential evaluation process is organized around an enhancement of the Smith set which is called the Schwartz set. This view of the procedure is of-

Figure 4: Beatpath for College Football

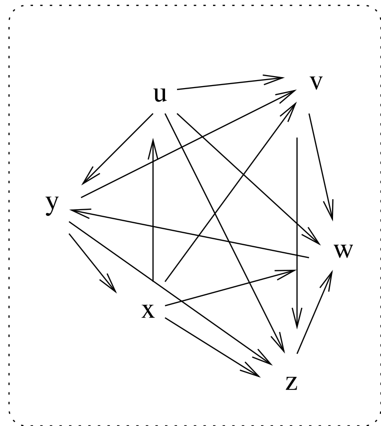


ten called Cloneproof Schwartz Sequential Dropping (CSSD). Recall the Smith set is the "top cycle" set of winners. The Smith set is not always selective; frequently it includes all of the alternatives. In order to be a member of the **Schwartz set**, an alternative must either be undefeated in pairwise competition (it may defeat *or tie* each other alternative) or it must be able to transitively defeat every other alternative that can transitively defeat it. To be in the Schwartz set, a candidate must have "electoral self defense" against each of the other candidates. More formally, x is in the Schwartz set if for any y such that $P[y, x] > 0$, then $P[x, y] > 0$. Even if x transitively defeats y, y can be in the Schwartz set if there is a chain of plurality decisions reaching from y to x. An alternative y is rejected from the Schwartz set if y cannot transitively defeat another alternative which can transitively defeat y. If there are no pairwise tie votes, the Schwartz set equals the Smith set. If there are ties, then the Schwartz set can be much smaller. In Figure 5, ties are represented by dotted lines that connect alternatives. The ties make a difference because the transitive defeat concept is based on pairwise victories and ties are treated the same as losses. In other words, ties break chains.

The sequential approach can be thought of as a way to find the strongest chains. The Schwartz sequential dropping version of the Schulze procedure is the one that is described in the Constitution of the Debian software organization `http://www.debian.org/vote/2003/vote_0002`. In Figure 5, we illustrate the sequential narrowing of the Schwartz set that results when ties (dotted lines) are inserted.

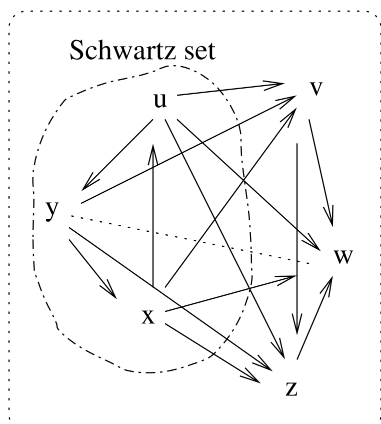
Figure 5: Schwartz Sequential Dropping

(a)Smith set = Schwartz set



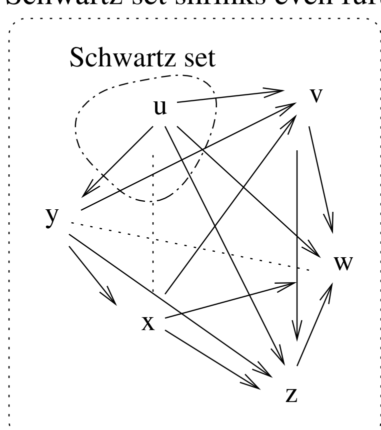
Smith set=Schwartz set

(b) One tie shrinks the Schwartz set



Smith set

(c) Schwartz set shrinks even further



Smith set

Schwartz sequential dropping proceeds iteratively, repeatedly narrowing down the alternatives. It is easy to make this sound more complicated than it really is, but we will try not to.

Step 1. Collect the ranking information from the voters. The results of all pairwise comparisons are known. Voters are allowed to rank all of the alternatives or just some of them. They are allowed to register their indifference between alternatives. From the ballots, we can calculate the pairwise outcomes for all alternatives, either $x \succ_P y$ or xI_Py or $y \succ_P x$. It is important to remember the number of votes for each alternative and the margins of victory.

Now the decision-making starts.

Step 2. Eliminate Rejects. Begin by simplifying the problem by isolating "rejects," alternatives that are losers in the round robin tournament. These losers are the ones that are left out of the Schwartz set. Recall that y is rejected if it is transitively defeated by some other proposal and there is no chain that leads from y to that other alternative. The rejects are eliminated from consideration, permanently.

Step 3. Evaluate the remaining contenders.

A. If there is only one contender, then it is the winner.

B. If there is more than one contender, check the pairwise comparisons among all of the contenders. If all pairs are tied, it means, in some sense, that we have found all of the alternatives that are "equally good." If all are tied, proceed to the tie-breaking procedure in Step 4.

C. If all contenders are not tied pairwise, then we need to make some changes so we can reject more alternatives. Here is where the "sequential dropping" kicks in. Among all pairs of contenders, find the "weakest pairwise defeat." (To us, a name like "least impressive victory" would be more fitting.) The idea is to spot the least worthy pairwise victor and demote it. How to spot the least impressive winner? Look for the alternative that wins with the smallest number of votes.[3] When the weakest pairwise defeat is found, that victory can be erased

[3]Some versions of this procedure use the margin of victory, rather than the absolute number of votes received by the winner. Schulze recommends the total number of votes because he feels it gives voters a positive incentive to report their rankings as completely as possible. Otherwise, they might have an incentive to report just the rankings of the top one or two alternatives in order to "game" the election system.

by changing it to a tie in the data. That is, we change the result in the round robin data for the weakest defeat from $\succ_P$ to $\approx_P$. (Caution: contrary to what some Web sites say, neither the weakest defeat nor the tied alternatives are not dropped literally. The weakest defeat is converted to a tie and the Schwartz set is recalculated.) If there are several alternatives that are equally weak winners–many weakest defeats exist–then choose the one with the most votes for the loser and declare that one to be the weakest defeat. If there are several pairs tied in both the number of votes for the winner and the loser, treat them all in the same way: change the pairwise comparison to a tie. This generates new round robin data.

Go back to Step 2 and repeat the procedure. Use the new round robin data, of course. The rejects that were eliminated in previous stages are never re-admitted. Since we have inserted some ties into the results, the members of the Schwartz set will change: some contenders will become rejects and then they are dropped. The procedure will repeatedly loop between Steps 2 and 3 until the condition described in Step 3B is obtained.

Step 4. If we arrived at this stage, it means we have narrowed down the alternatives to the most appealing set of more-or-less equally good outcomes. Schulze calls these "potential winners." A winner is selected by a tie breaking rule.

The tie breaking rule that is suggested by Schulze is, more-or-less, going to rank the tied outcomes by listening to the opinions of randomly drawn voters. Suppose the tied candidates are $\{w, x, y, z\}$. Randomly draw a ballot and copy down the preferences of the voter. If that ballot states a ranking for all of the alternatives, then that is the end of the procedure. The group result will match that ballot. If the ballot is only partially filled in, for example, it states only $x \succ_i y$, then we take that information and mark down $x \succ_{Schulze} y$ (it is "set in stone," unalterable) and we draw another ballot. If that ballot has preference information about candidates that are not already sorted by $\succ_{Schulze}$, such as w, and y, then use that information $w \succ_{Schulze} y$. We draw ballots at random until every pairwise comparison is filled in, and we *never* change any of the Schulze comparisons we have already completed. If we happen to go through all ballots without ranking all of the alternatives, then the final ranking is assigned at random.

The Schulze method can be applied to any of the voting examples we have considered. Of course, if the preferences imply the existence of a Condorcet Winner, then that alternative is chosen and the solution is simple. To find a more interesting case, we need to select an example in which there is a voting cycle. Recall the Borda paradox considered in section 4.2. In Figures 6 and 7, we have used the preferences from Tables 3 and 4 with the Schwartz sequential dropping approach.

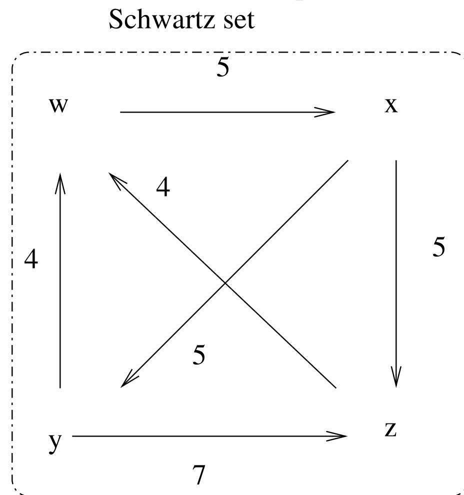
In Figure 6, we note that the original preferences allow a majority rule cycle across all of the alternatives, so the first stage Schwartz set includes all of the alternatives. There are two comparisons tied for the weakest defeat, each with the support of 4 voters, so in 6b, those arrows are changed to dotted lines representing ties. After those ties are taken into account, the Schwartz set is recalculated, and it includes only w.

Recall that alternative z is exposed as a fraud in that example. The effect of dropping z to the bottom of the voter preference orderings appears in 7. Note that the initial Schwartz set now includes only three alternatives, $\{w, x, y\}$. Alternative z is excluded (it is a Condorcet loser). The weakest defeat is between w and y. After that is changed to a tie, then w is the only potential winner remaining. The same alternative, w, wins with the Schulze method in both cases.

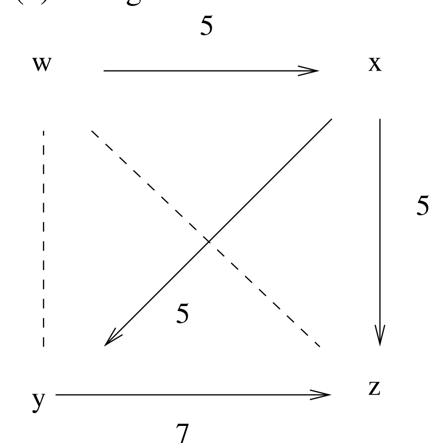
The Schulze method has the apparent strength of solving the paradox in the Borda count: w wins in both cases. The Schulze method is thus more stable and, in some sense, more logical. The fact that the Borda method chose y in the original example (Table 3) and Schulze chooses w deserves mention. From a beatpath point of view, it appears that we did not correctly understand the Borda problem in section 4.2. We thought the paradox was that y was turned from a winner into a loser when the fortunes of z changed. Now it appears that the original Borda winner, y, was "wrong." Borda chose the wrong winner because it allowed z to play the role of a spoiler, somehow distracting voters from their purpose, thus allowing y to win. When z is moved to the bottom in Table 4, its role (its influence as an irrelevant alterna-

Figure 6: Schulze Method Applied to Table 3

(a) Pairwise comparisons



(b) Change weakest defeats to ties



(c) Second-stage Schwartz set includes only w

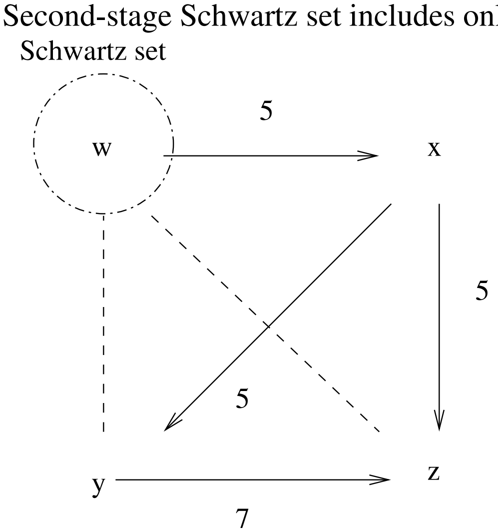
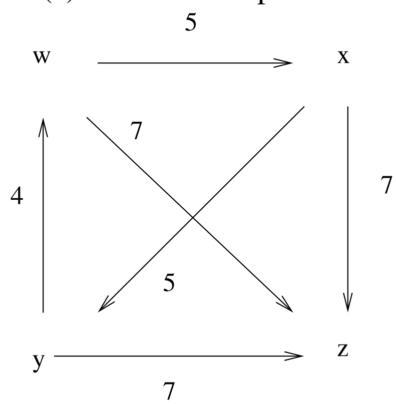
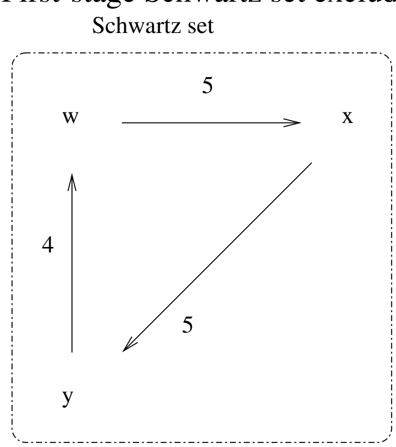


Figure 7: Schulze Method Applied to Table 4

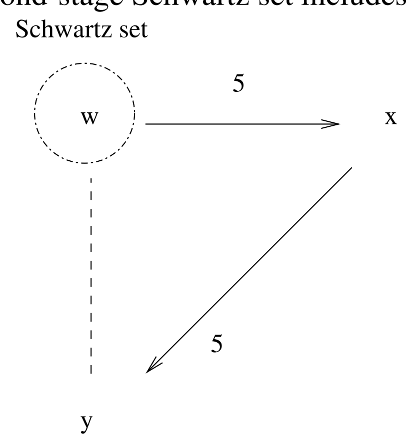
(a) Pairwise comparisons



(b) First-stage Schwartz set excludes z



(c) Second-stage Schwartz set includes only w



tive) is eliminated, and the "correct alternative" w wins.

The Schulze method leads to a full, transitive ordering of all of the candidates. The beat-path values $P[x, y]$ can be used to sort alternatives from top to bottom. As a result, the method could be used to determine the winner of the Division 1A College football championship on the basis of the rankings submitted by coaches, writers, or computers.

The reader might be wondering, "if this method is so great, why isn't everyone using it?" Well, if you ask its proponents, we expect they will say "because everyone is not very smart!" Ordinary people–not academics–are not particularly interested in election procedures, much less complicated ones. As a result, the most widely used argument against the method seems to be that it is too complicated. To academics, this is not a very good argument, but it carries a lot of weight with real life policy makers. In response, the Schulze proponents will point out that, from a voter's point of view, it is no more complicated than a Borda count. All the voters need to do is submit rankings of some alternatives. The counting process is more complex, of course, but the effort is justified. The proponents can list a long list of favorable qualities that we have already mentioned, plus some more.

The beatpath approach is not completely free from the anomalies that infect the other voting mechanisms. Schulze draws our attention to the paradoxical fact that his method violates the **Participation Criterion**. The participation criterion says that if an election has been decided, and then you add voters who prefer the winner to another alternative, then adding those new voters must not result in the victory of that other (less favored) alternative. More formally, the winner is x, hence socially $x \succ_{Schulze} y$. Add more voters such that $x \succ_i y$. Adding those voters should not change the social result to $y \succ_{Schulze} x$. The criterion does not require that x must still win, only that adding voters who do not prefer y should not make y the winner. Please note that the participation criterion is different from the monotonicity principle. Monotonicity states that *among existing voters*, a change in voter preferences that favors x against y must not hurt x's position in the final decision. Participation concerns the addition of new voters. Schulze is a monotonic procedure, but examples can be found to show that it violates the

Table 11: The Schulze Method

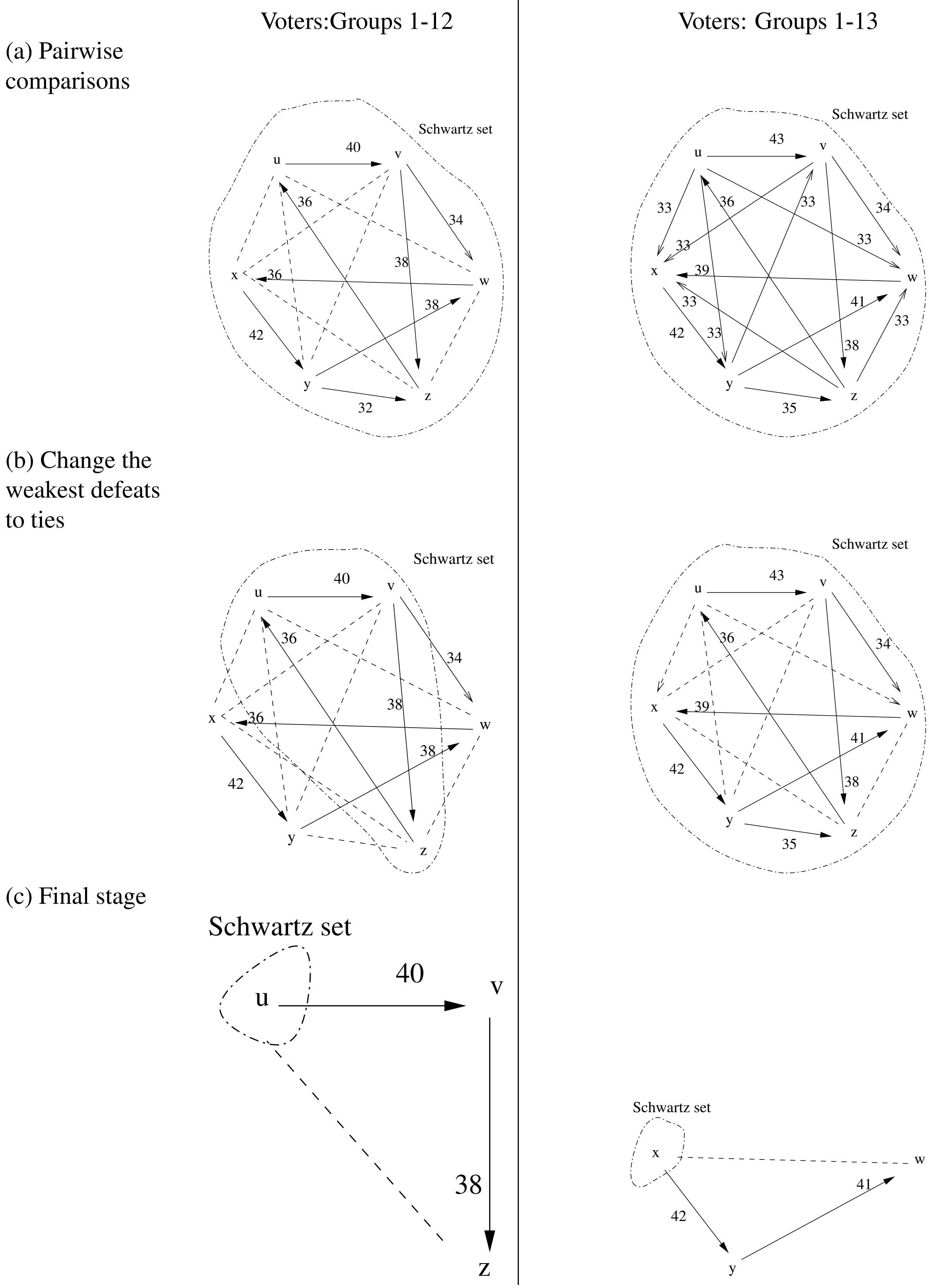
	Group:	1	2	3	4	5	6	7	8	9	10	11	12		13
	Number of voters:	4	2	4	2	2	2	4	12	8	10	6	4		3
	Preferences														
	favorite	u	u	u	u	v	w	w	x	y	z	z	z		u
	second	v	v	y	y	z	x	x	y	w	u	u	y		y
	third	w	z	v	z	u	v	v	w	x	v	v	x		z
	fourth	x	x	z	v	w	y	z	u	v	w	x	v		w
	fifth	y	y	w	w	x	z	y	v	z	x	y	w		v
	sixth	z	w	x	x	y	u	u	z	u	y	w	u		x

participation criterion.

Schulze's example of the violation of the Participation Criterion is presented in an example that has 6 candidates, $\{u, v, w, x, y, z\}$ (see Table 11). The winner is u when the electorate includes the 60 voters from groups 1-12 . The violation of the participation principle is shown when group 13, which has 3 members, is added to the electorate. For all of the members of group 13, note that u is the most preferred alternative and x is the least desirable. One would expect that their favorite, the winner u, would remain in good position. Unfortunately, the Schulze rule applied to this 63 voter electorate declares x to be the winner. After the sequential dropping process, u is the only contender (the only member of the final Schwartz set), no use of the "tie breaker" is required.

Among the collection of peculiar election outcomes that we have been accumulating, this is surely one of the most bizarre. In order to investigate it more carefully, consider the graphs that illustrate the sequential dropping process in Figure 8. On the left, the pairwise data for the 12 group problem is illustrated. Note there are many ties in this example. The weakest defeat is the step $y \succ_P z$ which is supported by 32 voters. After that is changed to a tie, the Schwartz set shrinks to $\{u, v, w\}$, and from there it is easy to drop the weakest defeat and see that u wins. On the right side of the figure, note that when group 13 is added, there are no pairwise ties. When the weakest defeats are changed to ties, then the pattern of ties and pairwise comparisons in the initial results on with 12 groups re-appear. The ties are the same, but the mar-

Figure 8: Schulze Method with Preferences in Table 11



gins of victory are different. With 13 groups, the weakest defeat is now $v \succ_P w$. After a tie is drawn between v and w, the Schwartz set shrinks to $\{x, w, y\}$, and from there, the result is easy to see: x wins. Why did u lose? When the path from v to w was removed, then u lost its chains of self defense. The alternatives x,y, and z can all transitively defeat u, and there is no path leading from u back to them if $v \approx_P w$.

This is a variant of the problem of independence from irrelevant alternatives. What is driving the problem here is not the relative evaluation of u and x, but rather the careful placement of u, v, w, and z in the orderings of group 13 voters. The careful placement ends up changing the magnitude of pairwise victories, thus impacting the order in which the sequential dropping eliminates candidates. If you think of the sequential dropping process as a series of games in a tournament, then, we have simply found another example of the way in which the ordering of the matches affects the outcome.

XX.6 Exercises

1. The following table describes the tastes of 3 voters. Conduct a round robin tournament and use the win-loss records to try to decide which candidate should win the tournament.

			Voters	
		1	2	3
Ranking				
first		x	y	z
second		y	z	x
third		z	x	y

2. Consider the table in the previous question. Describe a couple of changes you might make in voter preferences (either by changing voters or adding new ones) if z is to become the winner.

3. Using the preference table used in problem 1 of Section 4, conduct a round robin tournament and calculate the aggregated pairwise vote totals for the candidates. Confirm that the results are the same as the Borda count.

4. Consider the preferences used in problem 1 of Section 4. Create a graph of the type shown in Figure 4. Use that graph to figure out which candidate would win with the Schulze procedure.

5. Consider the preferences used in problem 2 of Section 4. Repeat the exercise of the previous question.

6. Use the graph you created in the previous question to figure out which alternatives are in the Schwartz set. Then change the weakest defeats to ties and re-calculate the Schwartz set.

7. This is a challenging problem, one which we refer to in the feature box on 42. Show that if the Borda count (or, equivalently, the aggregated pairwise vote) outcome is $x \succ_{AP} y \approx_{AP} z$, then the only pairwise outcomes that are consistent with that outcome are the ones listed in equation 8.

7 Single Vote Systems: Cousins of Plurality and Majority Rule

We began this chapter with the plurality problem. A plurality election with twenty or thirty candidates might award victory to a very unpopular candidate. The poster children for the problem might be the Democratic and Republican candidates in Minnesota's gubernatorial election who were edged-out by Jesse Ventura. In the words of mathematician Keith Devlin, Jesse Ventura "won not because the majority of the voters chose him, but because plurality voting effectively thwarted the will of the people. Had the voters been able to vote in such a way that, if their preferred candidate were not going to win, their preference between the remaining two would be counted, the outcome would have been quite different" (2004). The alternative schemes, the Borda rank order voting plan and the pairwise Condorcet systems, offer some hope. They have not won the hearts and minds of electoral reformers with the same power as the alternative to which we now turn our attention.

This system requires the voters to come to the polls only once. The voters rank the alternatives and then the election officials can proceed "as if" there really were a run-off election. When this procedure is used to select a single winner in an election, it is generally referred to as an Instant Runoff Vote (IRV) or an Alternative Vote system. When the method is used to select a group of candidates, such as the top 5 nominees for the Grammy Awards, or in a proportional legislative election, the procedure is called the Single Transferable Vote (STV). The procedures are the same for choosing a single winner, but the details get complicated for selecting the additional winners. There are two prominent implementations of the STV concept, the Hare system (in honor of Thomas Hare, an English proponent of this system), the Hare-Clark system (add an honorific for Andrew Inglis Clark, a Tasmanian Attorney-General who pushed a variant of the STV into usage). The transferable vote is in use in national elections in Australia, Malta and Ireland as well as in local elections in Cambridge, Massachusetts, San Francisco, California, and cities in Northern Ireland, and New Zealand. The method is used in faculty elections at Cornell University.

We will focus our attention on the selection of a single winner from a field of candidates, so we will detail only that part of the procedure.

The winner must obtain a target number of votes, dubbed the quota. If there is only one spot open–only one candidate can win–the usual quota is the majority of the votes:

$$quota = majority \geq \frac{Number\,of\,voters + 1}{2}$$

On election day, the voters go to the polls and rank the candidates from first to last in order of desirability. Then election managers name the winner by applying the following algorithm.

Step 1. A voter's support on the first ballot is given to the voter's top-ranked candidate. If one of the candidates has a majority, then that candidate is the winner. If no candidate has a majority, proceed to Step 2.

Step 2. The candidate with the smallest number of votes is eliminated from consideration. The ballots of the voters who had supported that eliminated candidate are reconsidered. Their votes are transferred to their second-ranked candidates. Then the votes are re-tabulated. If a candidate has a majority, then it is declared the winner. Otherwise, repeat Step 2. (The candidate that was eliminated in this step is excluded throughout the remainder of the process.)

If the election is choosing 5 members of a commission or a legislative delegation, the procedure is slightly more involved. Not only are the votes cast for the losers transferred, but also the votes received by the winners can be transferred. The details of this would take us too far away from the central purpose of this section, but we are able to describe the process in general terms. Candidates qualify for victory by exceeding a vote quota[4], a number smaller than a majority. If there is no candidate who meets the quota, then a loser is eliminated and votes are transferred to the second-ranked candidates. If a winning candidate has more votes than the quota requires–a "surplus" of votes–the surplus is reallocated to the next-best candidate on the surplus ballots. The calculations in that re-allocation are a bit involved; that is where the variants of STV diverge. The main idea in the procedure is that candidates are removed one-by-one, and the votes cast for the removed candidates are either counted for them (as part of their quota) or transferred and counted for other candidates that the voter also supports.

The proponents of this method claim that it does not throw away information in the same way as a plurality election. It saves voters from the terrible choice that is forced on them by a plurality vote. If the voters who favor either the Democrat or the Republican against Jesse Ventura have a chance to declare that priority, then Ventura might not end up winning in the Instant Runoff. The "Ventura problem," coupled with the on-going use of the procedure in

[4]
$$quota = \frac{Number\, of\, votes\, cast}{Number\, of\, winners\, to\, be\, chosen + 1} + 1$$
Note that if there is one winner to be chosen, the winner must have a bare majority.

some places, gives the STV advocates some persuasive ammunition. Krist Novoselic, most well known as a member of the band Nirvana, has written a book that combines his band memoir with an endorsement of transferable votes, *Of Grunge and Government: Let's Fix This Broken Democracy* (2004). He also hosts a website (http://www.fixour.us) where he posts editorials. He eagerly watched the introduction of STV in San Francisco for the 2004 Board of Supervisors elections. Steven Hill of the Center for Voting and Democracy, a proponent of the transferable vote systems, enthusiastically proclaimed, "More voters picked their supervisor, and fast results mean that San Francisco avoided a December runoff election, saving millions in taxes (the cost of the second election)"(Novaselic, 2004).

There is no doubt that the STV might work to correct bad outcomes in some particular examples. Perhaps Jesse Ventura might not have been governor of Minnesota. There are example problems that we can consider in which the STV does not seem to perform too badly. If the STV procedure is applied to the preferences of groups 1-12 in Table 11 (the data for the Schulze example that violated the participation criterion), the STV stages eliminate v, then w, then y, and then u. The two remaining candidates, x and z, are tied with 30 votes. After adding group 13 into the calculations, the elimination process rejects the alternatives in the same order (v, w, y, and then u). The winner is z, which earns 33 votes against the 30 for x. The new voters prefer z to x, and it seems quite reasonable that adding them to the electorate would tip the scales in favor of z.

On the other side of the ledger, however, the STV has several properties that should cause concern. The fact that the STV system does not always select a Condorcet Winner (when one exists) is a serious shortcoming. Academic research has often concentrated on another issue: STV is not monotonic. The principle of monotonicity has first come to our attention in the consideration of majority rule and May's Theorem. Now it comes to the forefront again. The basic idea of the Monotonicity Criterion is that an election system should be responsive, and not in a perverse way. If a voter changes his mind and decides to rank an alternative more highly, then that alternative's electoral fortunes should be helped, or at least not hurt. This is the property commonly called weak monotonicity.

Consider the preference rankings in Table 12. There are preferences for five groups, $\{1, 2, 3, 4, 5\}$ and four alternatives, $\{w, x, y, z\}$. The STV algorithm proceeds as follows.

1. The first place votes are, in order, w=9, x=6, y=6, z=5.

2. Eliminate z, so the candidates are w, x and y. Re-allocate the votes of group 1 to their second-ranked candidate, x. The votes are retabulated: x=11, w=9, y=6.

3. Eliminate y, so the final 2 candidates are w and x. Reallocate the votes of groups 2 and 3 to x. The final vote totals are x=17, w=9. So x is declared the winner.

We make only one small change in order to demonstrate the fact that STV is nonmonotonic. We will replace group 3 with the voters in group $3'$, so that the columns being used in the vote are $\{1, 2, 3', 4, 5\}$. We hope the reader will agree that this seems to be an innocuous change. The voters in 3 think x is second best, but the voters in $3'$ think that x is the most desirable of all; x should still win, shouldn't it?

Observe how the STV result changes.

1. The first place votes are, in order, w=9, x=8, y=4, z=5.

2. Eliminate y, so the candidates are w, x and z. Re-allocate the votes of group 2 to their second-ranked candidate, z. The votes are thus retabulated: z=9, w=9, x=8.

3. Eliminate z, so the final 2 candidates are z and w. Reallocate the votes of groups $3'$ and 4 from x to z. The final vote totals are z=17, w=9. The winner is z.

This is a clear violation of weak monotonicity (and, we might add, plain old common sense). The winner of the original STV was x, but after we raise x's popularity by replacing 3 with $3'$, the winner is z. Alternative x was eliminated in stage 3. This would be interpreted to mean that x finished in third place. And the only change between the first and second example was that the voters in group 3 became group $3'$, meaning they raised x in their preferences. After noting this fact, William Riker concluded "It is hard to believe that there is any good justification for the single transferable vote." (1982, p. 51).

Table 12: Single Transferable Vote Violates Monotonicity

			Voters					Alternative Preferences
	Groups	1	2	3	4	5		3’
Number of	Members	5	4	2	6	9		
Ranking								
first		z	y	y	x	w		x
second		x	z	x	y	z		y
third		y	x	z	z	x		z
fourth		w	w	w	w	y		w

XX.7 Exercises

1. Consider the preferences used in problem 1 of Section 4. Conduct an STV election. In what order will the candidates be eliminated? How will the votes be re-allocated? Who will be the eventual winner?

2. Repeat the previous exercise with the preferences used in problem 2 of Section 4.

3. Suppose you are a member of the book of the month club. The club must choose between the classics *The Grapes of Wrath* and *Superman: Battle to the Finish with Luther!* You would love to have *Superman*, but you know for sure that at least 70% of the club members are snobs who would vote for *Grapes*. Think of two methods you might try in order to bring about the eventual selection of *Superman.* Note, you are not allowed to make a persuasive speech to change their minds or do anything that would change the pairwise comparison of the two books, but you can introduce new alternatives and different voting methods.

8 Conclusion

After reviewing the many voting systems and studying the many principles by which they are judged, the reader has a right to ask the author, "how do you rank these systems? Which is the best?"

We find ourselves wishing that there were some voting system that would meet all of the requirements that seem fair and logical. Wishing does not make it so, sadly enough. In fact, there is a theorem, known as Arrow's Possibility Theorem, which states, basically, that it is mathematically impossible to find a voting system that satisfies what appear to be the most basic, bare-minimum requirements. Arrow, an economist, was awarded the Nobel prize, partly on the basis of this research (Arrow, 1963).

Arrow's theorem can be stated in several ways. We think this is the most useful version. Suppose there is an election ranking system that meets these requirements:

Nondictatorial The social ranking does not mirror the tastes of one voter i without regard to the tastes of the other voters.

Pareto Principle If all voters favor one alternative over another, then the social ranking should respect that and rank the alternatives accordingly.

These seem like such simple, fundamental elements that they can be assumed as necessary ingredients in a social ranking procedure. Arrow's Possibility Theorem says the following. If an election system is nondictatorial and obeys the Pareto principle, then it is possible to find an example set of voter preferences such that:

1. The social ranking is intransitive (a voting cycle), and/or
2. Independence of irrelevant alternatives (IIA) is violated.

It is very important to emphasize that the Possibility Theorem does not mean that a ranking procedure is always intransitive or in violation of IIA. Rather, it is always possible to find an example society, a set of voter preferences, such that the result is somehow illogical. If the result is intransitive, then we have the paradox of "rational man, irrational society"(Barry and

Hardin, 1982). If we adopt a system that blocks intransitivity, such as the Borda count, then we know for sure there are examples in which independence from irrelevant alternatives is violated.

Arrow's theorem is sometimes overstated. For example, some students think it says that democracy is always illogical. It does not mean anything of the kind. Rather, it means that it is not possible to find a democratic procedure that works in a sensible way all of the time.

If it is not possible to meet such basic, simple criteria as nondictatorship, of course it is not possible to meet even more elaborate properties like the participation criterion. As a result, the choice of electoral system has to be based not on which system meets all of the specified criteria, but rather on a value judgment about what sorts of peculiarities are the most harmful and which sorts of societal preference profiles are most likely to arise and cause peculiarities.

If the reader wants to know "which ranking system is best?" we can sympathize. We want to know the same thing. And so do most political scientists, economists, mathematicians, physicists, and psychologists.

Personally speaking, we have found a pleasant results by using procedures that eliminate the obviously bad candidates (by some principle like the Smith set or the Schwartz set) and then choosing among the remainder by a positional method like the Borda count or the Schulze method. We have been very unhappy with decisions reached by pure plurality or a runoff election. The STV might be preferable to a plurality or majority rule election, but the fact that it is so badly nonmonotonic causes us to shy away from it. That is to say, plurality rule is truly bad, as both Borda and Condorcet had contended, and we would rather have STV than plurality, but we'd rather have one of the more elaborate two-stage procedures instead of either one.

Suggested Readings

Readers who look into this further will no doubt find ten or twenty seriously proposed election procedures, all of which have advantages worth considering. Perhaps one would begin with the textbook by by William Riker (1982) and then do some soaking and poking in the Internet. We have found many fascinating Web sites and would encourage readers to adventure, but keep in mind that these sites are not peer reviewed by experts and we have noticed a non-trivial number of mistakes in interpretation and mathematics. Perhaps the most rigorous analysis is to be found on the election-methods email list, which is hosted on http://electorama.com. In the next two sections, we discuss two of the procedures that have been recently been proposed and have well-documented merit.

Exercises

Answer to question XX.6.7.

We want to show that if $x \succ_{BC} y \approx_{BC} z$, then the valid pairwise outcomes are:

$$\begin{array}{ll} 1. & x \succ_P y,\ x \succ_P z,\ y \approx_P z \\ 2. & x \succ_P y, x \succ_P z, y \succ_P z \\ 3. & x \succ_P y, z \succ_P x, y \succ_P z \\ 4. & x \succ_P y, x \approx_P z, y \succ_P z \\ 5. & x \approx_P y, x \succ_P z, y \succ_P z \end{array} \tag{10}$$

Recall that the none of the voters are indifferent between alternatives. Use the shorthand notation $q_{x,y}$ to represent the number of voters who prefer x to y.

$$q_{x,y} = |\{i \in N : x \succ_i y\}|.$$

Because there is no indifference, then a tie means $q_{x,y} = q_{y,x} = N/2$. If $q_{x,y} > q_{y,x}$, then x defeats y in a head-to-head comparison (equivalently, $q_{x,y} - q_{y,x} > 0$ means $x \succ_P y$). The results of the aggregated pairwise vote, which is equivalent to the Borda count, can be expressed as

$$aggregated\, pairwise\, vote\, for\, x:\ q_{x,y} + q_{x,z}$$

We calculate that aggregates (equivalently, Borda Counts) for y and z as well

$$aggregated\, pairwise\, vote\; for\, y:\; q_{y,x} + q_{y,z}$$

$$aggregated\, pairwise\, vote\; for\, z:\; q_{z,x} + q_{z,y}$$

If the Borda outcome is $x \succ_{BC} y \approx_{BC} z$, then it must be true that:

$$\begin{array}{rcl} q_{x,y} + q_{x,z} & > & q_{y,x} + q_{y,z} \\ q_{x,y} + q_{x,z} & > & q_{z,x} + q_{z,y} \\ q_{y,z} + q_{y,x} & = & q_{z,x} + q_{z,y} \end{array} . \tag{11}$$

Keep in mind that $q_{y,x} = N - q_{x,y}$,$q_{z,x} = N - q_{x,z}$, and $q_{z,y} = N - q_{y,z}$; although it appears there are six variables in these expressions, there are really just three unknowns.

Re-arrange the inequalities in 11 so that they have the pairwise majority rule conditions on the left hand side:

$$\begin{array}{rcl} q_{x,y} - q_{y,x} & > & q_{y,z} - q_{x,z} \\ q_{x,z} - q_{z,x} & > & q_{z,y} - q_{x,y} \\ q_{y,z} - q_{z,y} & = & q_{z,x} - q_{y,x} \end{array} \tag{12}$$

To find all sets of pairwise outcomes that are consistent with $x \succ_{BC} y \approx_{BC} z$, consider 3 cases, $y \approx_P z$, $y \succ_P z$, and $z \succ_P y$.

Case 1: $y \approx_P z$. This is a majority rule tie between y and z. Are there pairwise outcomes for x against y implied by this outcome? If $y \approx_P z$, then x defeats both y and z in pairwise contests. By definition, $y \approx_p z$ means $q_{y,z} - q_{z,y} = 0$, and since $q_{z,y} = N - q_{y,z}$, then $q_{y,z} = q_{z,y} = N/2$. The second condition in 12 reduces to:

$$q_{x,z} - q_{z,x} > \frac{N}{2} - q_{x,y} \tag{13}$$

The third condition in 12 also implies the restriction $q_{z,x} = q_{y,x}$. That means the support in favor of z against x must be exactly equal to the support for y against x, so $q_{x,y} = q_{x,z}$. Substitute $q_{z,x}$ in place of $q_{x,y}$ in 13 and we find:

$$q_{x,z} > \frac{N}{2}$$

This means that $x \succ_P z$ and $x \succ_P y$ are implied by the condition $y \approx_P z$, so one word has to be the first one listed in 10. That is to say, in order to be consistent with the Borda outcome $x \succ_{BC} y \approx_{BC} z$, then if y ties z in a plurality election, then x must defeat both y and z (and by the same margin).

Case 2: $y \succ_P z$. Alternative y defeats z in a pairwise majority contest. It is difficult to understand how it is possible for this to happen if $y \approx_{BC} z$. Beginning with the information supplied by the Borda outcomes, we can use the identities $q_{x,y} = N - q_{y,x}$, $q_{y,z} = N - q_{z,y}$,and $q_{z,x} = N - q_{x,z}$ to arrive at the following:

$$\begin{aligned} q_{x,y} - \tfrac{N}{2} &> \tfrac{1}{2}q_{y,z} - \tfrac{1}{2}q_{x,z} \\ q_{x,z} - \tfrac{N}{2} &> \tfrac{N}{2} - \tfrac{1}{2}q_{y,z} - \tfrac{1}{2}q_{x,y} \\ q_{y,z} &= \tfrac{N}{2} - \tfrac{1}{2}q_{x,z} + \tfrac{1}{2}q_{x,y} \end{aligned} \tag{14}$$

Insert the value of $q_{y,z}$implied by the third expression in the second one to obtain

$$q_{x,z} - \frac{N}{2} > \frac{N}{4} + \frac{1}{4}q_{x,z} - \frac{3}{4}q_{x,y}$$

which can be rearranged as:

$$\frac{3}{4}q_{x,z} - \frac{3N}{4} > -\frac{3}{4}q_{x,y},$$

or

$$q_{x,z} + q_{x,y} > N \tag{15}$$

We do not know for sure if x defeats y or x defeats z in pairwise votes, but we can say for sure that x will defeat at least one (possibly both) of them.

That seems like pretty thin gruel, but it actually helps us to narrow down the possibilities quite a bit. Since we assumed $y \succ_P z$, then 15 implies that the only majority rule outcomes possibly consistent with the Borda result $x \succ_{BC} y I_{BC} z$ are:

$$\begin{array}{ll} 1 & x \succ_P y, x \succ_P z, y \succ_P z \\ 2 & y \succ_P x, x \succ_P z, y \succ_P z \\ 3 & x \succ_P y, z \succ_P x, y \succ_P z \\ 4 & x \approx_P y, x \succ_P z, y \succ_P z \\ 5 & x \succ_P y, x \approx_P z, y \succ_P z \end{array} \tag{16}$$

Possibilities 2 and 4 can be ruled out. If $y \succ_P z$, it must also be true that $z \succ_P x$. To see why, recall $x \succ_{BC} y \approx_{BC} z$, so it must be that

$$q_{y,z} + q_{y,x} = q_{z,x} + q_{z,y}$$

Equivalently,

$$q_{y,z} - q_{z,y} = q_{z,x} - q_{y,x}$$

If $y \succ_P z$, $q_{y,z} - q_{z,y} > 0$ and so $q_{z,x} - q_{y,x} > 0$ and using the identities like $q_{x,y} + q_{y,x} = N$, we find $q_{x,y} - q_{x,z} > 0$. If $x \succ_P z$, then $q_{x,z} > \frac{N}{2}$, and it must be that $q_{x,y} > \frac{N}{2}$ (equivalently, $x \succ_P y$). The possibilities 2 and 4 violate that restriction. That leaves possibilities 1, 3, and 5, which correspond to three of the words in the solution.

Case 3: $z \succ_P y$. If we repeat the same exercise as the previous case, we arrive at the same set of possible words, except that the last one "swaps" the positions of y and z .

$$\begin{array}{ll} 1 & x \succ_P y, x \succ_P z, y \succ_P z \\ 3 & x \succ_P y, z \succ_P x, y \succ_P z \\ 5 & x \approx_P y, x \succ_P z, y \succ_P z \end{array} \tag{17}$$

Notice that two of these solutions are identical to the ones found in the previous paragraph, only the third one is unique. Collecting up the unique solutions, we have finished our work. All five of the words that are compatible with the Borda outcome $x \succ_{BC} y \approx_{BC} z$ have been found.

References

Arrow, Kenneth. 1963. *Social Choice and Individual Values, 2ed*. Wiley.

Barry, Brian W. and Russell Hardin. 1982. *Rational Man and Irrational Society? An Introduction and Sourcebook*. Beverly Hills: Sage Publications.

Bauer, Alissa. December 8, 2004. "Rank System Strips Teams of Postseason Opportunities." *University Daily Kansan* .

Black, Duncan. 1958. *The Theory of Committees and Elections*. Cambridge: Cambridge University Press.

de Borda, Jean-Charles. 1781. *Mémoire sur les Élections au Scrutin'*. Paris: Histoire del l'Académie Royale des Sciences.

de Condorcet, Marquis. 1785. *Éssai sur L'application L'analyse à la probabilité des dés Décisions Rendues à la Pluralité des Voix (trans. Essay on the Application of Mathematics to the Theory of Decision-Making)*. Paris.

Devlin, Keith. 2004. "Election Math." *Devlin's Angle, MAA Online* .
URL `http://www.maa.org/devlin/devlin_114_04.html`

Fishburn, Peter C. 1974. "Paradoxes of Voting." *American Political Science Review* 68(2):537–556.

Harsanyi, John C. 1955. "Cardinal Welfare, Individualistic Ethics, and Interpersonal Comparisons of Utility." *Journal of Political Economy* 61:309–321.

Hilburn, Robert. February 28, 2002. "It's Time to Nominate a New System." *Los Angeles Times* .

Johnson, Paul E. 1998. *Social Choice: Theory and Research*. Thousand Oaks, CA: Sage.

Jones, Bradford, Benjamin Radcliffe, Charles Taber, and Richard Timpone. 1995. "Condorcet Winners and the Paradox of Voting." *American Political Science Review* 89(1):137–144.

Mackenzie, Dana. 2000a. "Making Sense Out of Consensus: How Did That Guy Win?" *SIAM News* 33.

Mackenzie, Dana. 2000b. "May the Best Man Lose." *Discover* .

May, Kenneth O. 1952. "A Set of Independent Necessary and Sufficient Conditions for Simple Majority Decision." *Econometrica* 20:680–784.

Nash, John F. 1950. "The Bargaining Problem." *Econometrica* 18:155–162.

Nelson, Randy A., Michael R. Donihue, Donald M. Waldman, and Calbraith Wheaton. 2001. "What's An Oscar Worth?" *Economic Inquiry* 39(1):1–16.

Novaselic, Krist. 2004. "Instant Runoff Voting Used for First Time in San Francisco Elections." http://www.fixour.us/editorials/11.08.04.html.
URL `http://www.fixour.us/editorials/11.08.04.html`

Novoselic, Krist. 2004. *Of Grunge & Government: Let's Fix This Broken Democracy!* Akashic Books.

Reilly, Benjamin. 2002. "Social Choice in the South Seas: Electoral Innovation and the Borda Count in the Pacific Island Countries." *International Political Science Review* 23(4):355–372.

Riker, William H. 1982. *Liberalism Against Populism: A Confrontation Between the Theory of Democracy and the Theory of Social Choice*. San Francisco, CA: W. H. Freeman.

Riker, William H. 1986. *The Art of Political Manipulation*. New Haven, CT: Yale University Press.

Saari, Donald G. 1994. *Geometry of Voting*. Berlin: Springer-Verlag.

Schulze, Markus. 2003. “A New Monotonic and Clone-Independent Single-Winner Election Method.” *Voting Matters* 17:9–19.

Woodling, Chuck. December 24, 2004. “Maybe It’s Time To Do Away with AP Poll.” *Lawrence Journal World* page 3C.