A Comparison of Complementary and Kalman Filtering

WALTER T. HIGGINS, JR.
Member, IEEE
Arizona State University
Tempe, Ariz. 85281

Abstract

A technique used in the flight control industry for estimation when combining measurements is the complementary filter. This filter is usually designed without any reference to Wiener or Kalman filters, although it is related to them. This paper, which is mainly tutorial, reviews complementary filtering and shows its relationship to Kalman and Wiener filtering.

Manuscript received August 6, 1974; revised December 27, 1974.

I. Introduction

A simple estimation technique that is often used in the flight control industry to combine measurements is the complementary filter [1]. This filter is actually a steady-state Kalman filter (i.e., a Wiener filter) for a certain class of filtering problems. This relationship does not appear to be well known by many practitioners of either complementary or Kalman filtering. One exception is the tutorial paper by Brown [2] which discusses this relationship without going into the mathematical details.

The complementary filter users do not consider any statistical description for the noise corrupting the signals, and their filter is obtained by a simple analysis in the frequency domain. The proponents of the Kalman filtering approach work in the time domain and do not pay much attention to the transfer function or frequency domain (Wiener filter) approach to the filtering problem, since it is a less general approach to the filtering problem. The Wiener filter solution to this class of multiple-input estimation problems appeared in the literature [3], [4] well before Kalman published his classic paper [5].

This paper reviews complementary filtering and shows how this technique is related to Kalman and Wiener filtering. Since both Kalman and complementary filtering are under consideration for use in the Space Shuttle Reentry and Landing Navigation System, the relationship between them should be well understood.

II. Complementary Filtering

The basic complementary filter is shown in Fig. 1(A) where x and y are noisy measurements of some signal z and $\hat{z}$ is the estimate of z produced by the filter. Assume that the noise in y is mostly high frequency, and the noise in x is mostly low frequency. Then $G(s)$ can be made a low-pass filter to filter out the high-frequency noise in y. If $G(s)$ is low-pass, $[1 - G(s)]$ is the complement, i.e., a high-pass filter which filters out the low-frequency noise in x. No detailed description of the noise processes are considered in complementary filtering.

The complementary filter can be reconfigured as in Fig. 1(B). In this case the input to $G(s)$ is $y - x = n_2 - n_1$, so that the filter $G(s)$ just operates on the noise or error in the measurements x and y. Note that, in the case of noiseless or error-free measurements, $\hat{z} = z[1 - G(s)] + zG(s) = z$; i.e., the signal is estimated perfectly.

A typical application of the complementary filter is to combine measurements of vertical acceleration and barometric vertical velocity to obtain an estimate of vertical velocity. To fit the previous discussion, assume that the acceleration measurement is integrated to produce a velocity signal $\dot{h}_a$, as shown in Fig. 2. The integration attenuates the high-frequency noise in the acceleration measurement, whereas the noise in $\dot{h}_b$ is not changed. Therefore, if h_b is filtered by the low-pass filter

$$G(s) = 1/(\tau s + 1), \tag{1}$$

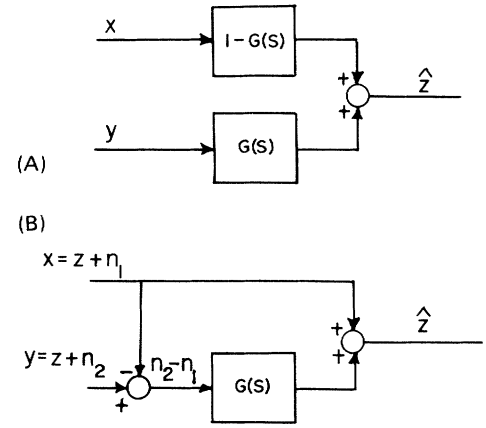
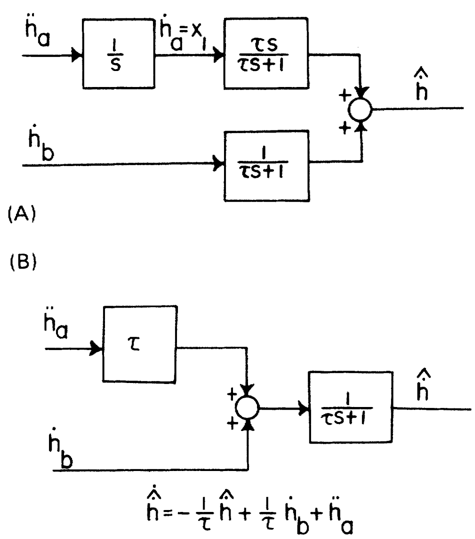


Fig. 1. (A) Basic complementary filter. If $G(s)$ is a low-pass filter, $1 - G(s)$ is a high-pass filter. (B) Alternate version of the filter in which the filter operates only on the noise.

Fig. 2. Complementary filter for estimating vertical velocity. (A) Basic complementary filter. (B) Actual realization of the filter.



then $\dot{h}_a$ is filtered by the high-pass filter

$$1 - G(s) = 1 - 1/(\tau s + 1) = \tau s/(\tau s + 1). \tag{2}$$

The filter of Fig. 2(A) can be simplified, and the actual realization of the filter is shown in Fig. 2(B). The time constant τ is usually between 2 and 6 seconds and is adjusted during simulation or flight testing. Note that the measurement $\ddot{h}_a$ is actually low-pass filtered even though $\dot{h}_a$ is high-pass filtered.

In the case of an augmented inertial system, an acceleration measurement is combined with a position measurement, and position and velocity are estimated. Fig. 3 shows how the complementary filter approach can be used to solve this problem. Fig. 3(A) illustrates the complementary filter which estimates the velocity from position and acceleration measurements. $G_1(s)$ must be a second-order transfer function in order for the transfer function from

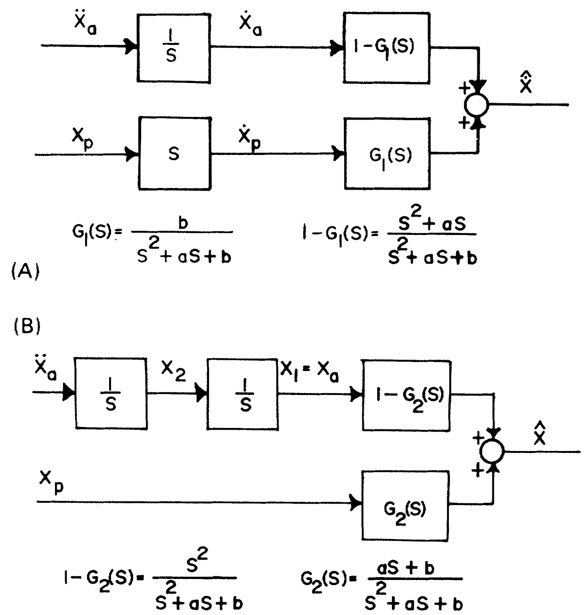


Fig. 3. Complementary filters to estimate (A) velocity and (B) position from acceleration and position measurements.

x_p to $\hat{\dot{x}}$ to provide attenuation at high frequencies. $G_1(s)$ must also have unity gain at low frequencies. Fig. 3(B) shows the complementary filter for estimating position from the position and acceleration measurements. In this case, in order to have a second-order transfer function between $\ddot{x}_a$ and $\hat{x}$, $1 - G_2(s)$ must be of the form

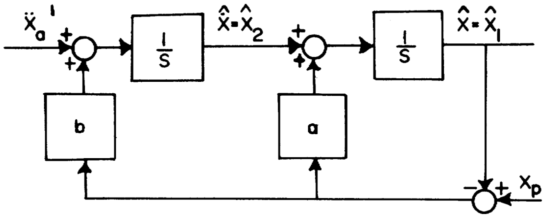
$$1 - G_2(s) = s^2/(s^2 + as + b). \tag{3}$$

Again, $G_2(s)$ has unity gain at low frequencies. The parameters a and b can be chosen to give the filter some desired natural frequency and damping factor.

Fig. 4 shows the actual realization of the filter. This multiple-input/multiple-output system can be realized by just the simple second-order system. The transfer functions from $\ddot{x}_a$ and x_p to $\hat{x}$ and $\hat{\dot{x}}$ are the same as in Fig. 3(A) and (B). This version of the filter also can be obtained by a direct argument as follows. The acceleration measurement is integrated to produce a velocity estimate and a position estimate. The position estimate is differenced with the position measurement to produce an error signal which is fed back to produce corrections in the estimates.

III. The Kalman Filter

Kalman filters, as they are used in navigation systems, are based on the complementary filtering principle. Brown, in his paper, refers to this as the complementary constraint. The basic block diagram is given in Fig. 5, although, as in the previous cases, the actual implementation may be different. Note the similarity between Fig. 5 and Fig. 1(B). The complementary constraint means that the filter just operates on the noise and is not affected by



$$\begin{bmatrix}\dot{\hat{x}}_1\\ \dot{\hat{x}}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}\hat{x}_1\\ \hat{x}_2\end{bmatrix} + \begin{bmatrix}a\\ b\end{bmatrix}(x_p - \hat{x}_1) + \begin{bmatrix}0\\ 1\end{bmatrix}\ddot{x}_a$$

Fig. 4. Actual implementation and equations of complementary filter to estimate position and velocity.

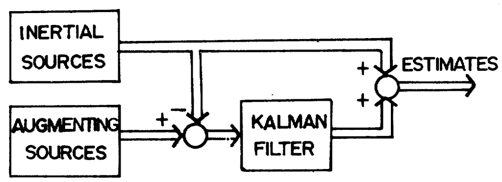


Fig. 5. Typical application of the Kalman filter in inertial navigation [2].

actual signals that are to be estimated. The advantages and disadvantages of removing this constraint are discussed by Brown.

In applying Kalman filtering to the problem of combining noisy measurements, the philosophy used is that the processing of one class of measurements defines the basic process equations. The other measurements, sometimes referred to as augmenting measurements, define the measurement equations for the filter. After discussing the basic equations, the two examples of the previous section are reworked using the steady-state Kalman filter approach. These examples can also be solved by the Wiener filter approach using spectrum factorization. The relationship between the steady-state or stationary Kalman filter and the Wiener filter is discussed in the book by Sage and Melsa [6].

Basically, there are two measurements, one of which serves as an input to a differential equation which serves as the process model. The ideal equations are

$$\dot{\boldsymbol{x}}_I = F\boldsymbol{x}_I + \boldsymbol{g}u \qquad \text{(process)} \tag{4}$$

$$z_I = \boldsymbol{h}\boldsymbol{x}_I \qquad \text{(measurement)} \tag{5}$$

where u is one noiseless measurement and z_I is the other. F, $\boldsymbol{g}$, $\boldsymbol{h}$, and $\boldsymbol{x}$ are $n \times n$, $n \times 1$, $1 \times n$, and $n \times 1$ matrixes, respectively; z_I and u are scalars. In actuality, we have two noisy measurements, so that the equations are

$$\dot{\boldsymbol{x}} = F\boldsymbol{x} + \boldsymbol{g}(u + w) \tag{6}$$

$$z = \boldsymbol{h}\boldsymbol{x}_I + v \tag{7}$$

where w and v are zero-mean, white, Gaussian noise.

The error equations are

$$\delta\boldsymbol{x} = \boldsymbol{x} - \boldsymbol{x}_I \tag{8}$$

$$\delta\dot{\boldsymbol{x}} = \dot{\boldsymbol{x}} - \dot{\boldsymbol{x}}_I = F\boldsymbol{x} + \boldsymbol{g}u + \boldsymbol{g}w - F\boldsymbol{x}_I - \boldsymbol{g}u$$

$$\delta\dot{\boldsymbol{x}} = F\delta\boldsymbol{x} + \boldsymbol{g}w \tag{9}$$

$$\delta z = z - \boldsymbol{h}\boldsymbol{x} = -\boldsymbol{h}\delta\boldsymbol{x} + v = \boldsymbol{h}_1\delta\boldsymbol{x} + v \tag{10}$$

where $\delta\boldsymbol{x}$ is the error vector.

The Kalman filter equation is [7]

$$\delta\dot{\hat{\boldsymbol{x}}} = F\delta\hat{\boldsymbol{x}} + \boldsymbol{k}[\delta z - \boldsymbol{h}_1\delta\hat{\boldsymbol{x}}] \tag{11}$$

where $\delta\hat{\boldsymbol{x}}$ is the estimate of the error vector and $\boldsymbol{k}$ is the Kalman filter gain. $\boldsymbol{k}$, an $n \times 1$ matrix, is obtained from the equations

$$\boldsymbol{k} = P\boldsymbol{h}_1{}^T R^{-1} = -P\boldsymbol{h}^T R^{-1} \tag{12}$$

where P, the $n \times n$ error covariance matrix, is the solution of the Riccati equation

$$\dot{P} = FP + PF^T - P\boldsymbol{h}_1{}^T R^{-1}\boldsymbol{h}_1 P + \boldsymbol{g}Q\boldsymbol{g}^T \tag{13}$$

in which $R = \sigma_v^2$ is the variance of the measurement noise and $Q = \sigma_w^2$ is the variance of the process noise. The stationary Kalman filter is obtained by setting $\dot{P} = 0$ in the Riccati equation. The actual estimates of the signals are

$$\hat{\boldsymbol{x}} = \boldsymbol{x} - \delta\hat{\boldsymbol{x}}. \tag{14}$$

In order to show the relationship with the complementary filters, the above equations can be manipulated to produce a differential equation for $\hat{\boldsymbol{x}}$ directly:

$$\dot{\hat{x}} = \dot{x} - \delta\dot{\hat{x}}$$
$$\dot{\hat{x}} = Fx + g(u + w) - F\delta\hat{x} - k[\delta z - \boldsymbol{h}_1\delta x].$$
But $\hat{x} = x - \delta x$, $\delta x = x - \hat{x}$, and $\boldsymbol{h}_1 = -\boldsymbol{h}$, so that
$$\dot{\hat{x}} = F\hat{x} + g(u + w) - k[z - \boldsymbol{h}x + \boldsymbol{h}(x - \hat{x})]$$
$$\dot{\hat{x}} = F\hat{x} + g(u + w) - k[z - \boldsymbol{h}\hat{x}]. \tag{15}$$

As is shown below, this equation is identical to the differential equations of the complementary filters for the two examples under consideration.

Example 1

The process equation from Fig. 2(A) is

$$\dot{x}_1 = \dot{h}_a = Fx_1 + g\ddot{h}_a = Fx_1 + g(\ddot{h} + w)$$

$$z = \dot{h}_b = \dot{h} + v.$$

Therefore, $F = 0$, $g = 1$, and $\boldsymbol{h} = 1$, so that the algebraic Riccati equation is

$$-PR^{-1}P + Q = 0$$

or

$$P = \sqrt{RQ} = \sigma_v\sigma_w$$

and

$$k = -\sigma_v \sigma_w / \sigma_v^2 = -\sigma_w / \sigma_v.$$

The filter equation is obtained by substituting into (15):

$$\dot{\hat{x}} = \ddot{h}_{acc} + (\sigma_w/\sigma_v)\,[\dot{h}_b - \hat{x}]$$

$$\dot{\hat{x}} = (-\sigma_w/\sigma_v)\,\hat{x} + (\sigma_w/\sigma_v)\dot{h}_b + \ddot{h}_a. \qquad (16)$$

This equation is identical to the equation of the complementary filter in Fig. 2(B), where the time constant of the filter is now $\tau = \sigma_v/\sigma_w$. Note that a time constant of four, as in the complementary filter, means that the barometric signal is assumed to be much noisier than the accelerometer signal. In the complementary filter, the time constant is chosen to get most of the information from the accelerometer signal and use the barometric information only as a long-term reference.

Example 2

The process equation from Fig. 3(B) is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \ddot{x}_a \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (\ddot{x} + w)$$

$$z = x_p = x_1 + v = [1\ 0]\boldsymbol{x} + v.$$

Therefore,

$$F = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad g = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad h = [1\ \ 0].$$

The solution to the algebraic Riccati equation is

$$p_{11} = \sqrt{2\sigma_w \sigma_v^3}$$

$$p_{12} = \sigma_w \sigma_v$$

$$p_{22} = \sigma_w \sqrt{2\sigma_v}$$

and the Kalman gain is

$$k = -P \begin{bmatrix} 1 \\ 0 \end{bmatrix} [\sigma_v^2]^{-1} = - \begin{bmatrix} \sqrt{2\sigma_w/\sigma_v} \\ \sigma_w/\sigma_v \end{bmatrix}.$$

The filter equation is

$$\dot{\hat{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \hat{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \ddot{x}_a + \begin{bmatrix} \sqrt{2\sigma_w/\sigma_v} \\ \sigma_w/\sigma_v \end{bmatrix} (x_p - \hat{x}_1). \qquad (17)$$

This equation is identical to the complementary filter of Fig. 3(C) if a and b of the complementary filter are set equal to k_1 and k_2 of the Kalman filter. Therefore, the assumption that the measurements are corrupted by stationary white noise produces a stationary Kalman filter that is identical in form to the complementary filter.

IV. Digital Implementation

Since modern inertial navigation systems use digital computers, the continuous filters can be replaced by discrete approximations, or the problem can be formulated as a sampled measurement problem from the start. The complementary or stationary Kalman filter has a considerable advantage over the normal Kalman filter because the Riccati equation and Kalman gains are not computed. Therefore, the update rate of the complementary filter can be higher than the normal Kalman filter. This is an important consideration in the applications to automatic landing problems, especially in an unpowered vehicle, such as the space shuttle, which has a rapid descent rate before final flare.

One simple method to obtain discrete equations is to replace the integrators in the block diagrams by digital integrators. Another method is to obtain difference equations directly from the differential equations of the filter. Consider the solution to the differential equation (17) from one sample time to the next:

$$\hat{x}(nT) = e^{FT}\boldsymbol{x}[(n-1)T] + \int_{(n-1)T}^{nT} e^{F(t-\tau)}(k\Delta x + g\ddot{x}_a)\, d\tau$$

where the state transition matrix is

$$e^{FT} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}$$

and

$$\Delta x(t) = x_p(t) - \hat{x}_1(t).$$

Assuming that T is small, $\Delta x(t)$ and $\ddot{x}_a(t)$ can be assumed constant over the sampling interval, so that the integral becomes

$$\int_{(n-1)T}^{nT} \begin{bmatrix} 1 & nT-\tau \\ 0 & 1 \end{bmatrix} d\tau\, (k\Delta x_{n-1} + g\ddot{x}_{a_{n-1}})$$

$$= \begin{bmatrix} T & \frac{T^2}{2} \\ 0 & T \end{bmatrix} (k\Delta x_{n-1} + g\ddot{x}_{a_{n-1}})$$

$$\simeq \begin{bmatrix} 1 & \frac{T}{2} \\ 0 & 1 \end{bmatrix} kT\Delta x_{n-1} + \begin{bmatrix} T/2 \\ 1 \end{bmatrix} \Delta v_{x_{n-1}}$$

where $T\ddot{x}_a \simeq \Delta v_x$. Δv_x is the usual output of an inertial measurement unit. Therefore, the final set of difference equations is

$$\hat{x}_n = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \hat{x}_{n-1} + \begin{bmatrix} 1 & T/2 \\ 0 & 1 \end{bmatrix} kT\Delta x_{n-1} + \begin{bmatrix} T/2 \\ 1 \end{bmatrix} \Delta v_{x_{n-1}}.$$

V. Conclusions

The relationship between the complementary filter and the Kalman filter has been shown. The complementary filter is simpler because it involves less computation. The question that remains to be answered is how does the accuracy of the two techniques compare? Does the use of fixed or preprogrammed gains degrade the filter performance significantly? In idealized cases, as the examples in this paper, the mean-squared error for given white-noise inputs can be compared. However, in a specific real-world problem, the noise is not really white, the position measurement is a nonlinear function of certain ranges and angles, and the filter equations are higher order, since there are three positions and velocities to be determined. A true comparison of the two filters would probably involve an extensive Monte Carlo simulation.

References

[1] S.S. Osder, W.E. Rouse, and L.S. Young, "Navigation, guidance and control systems for V/STOL aircraft," *Sperry Tech.* vol. 1, no. 3, 1973.

[2] R.G. Brown, "Integrated navigation systems and Kalman filtering: a perspective," *Navigation, J. Inst. Navigation,* vol. 19, no. 4, pp. 355-362, Winter 1972-73.

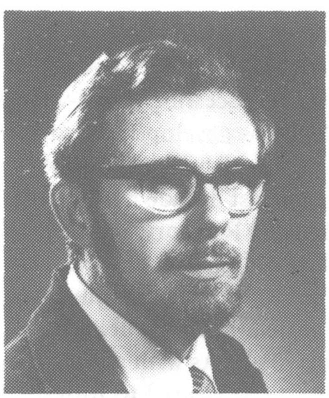
[3] R.M. Stewart and R.J. Parks, "Degenerate solutions and algebraic approach to the multiple input linear filter design problems," *IRE Trans. Circuit Theory,* vol. CT-4, pp. 10-14, 1957.

[4] J.S. Bendat, "Optimum filters for independent measurements of two related perturbed messages," *IRE Trans. Circuit Theory,* vol. CT-4, pp. 14-19, 1957.

[5] R.E. Kalman, "A new approach to linear filtering and prediction problems," *Trans. ASME, J. Basic Engrg.,* vol. 82D, pp. 34-45, March 1960.

[6] A.P. Sage and J.L. Melsa, *Estimation Theory with Applications to Communications and Control.* New York: McGraw-Hill, 1971.

[7] J.S. Meditch, *Stochastic Optimal Linear Estimation and Control.* New York: McGraw-Hill, 1969.



Walter T. Higgins, Jr. (S'60–M'66) was born in the Bronx, N.Y., on December 24, 1938. He received the B.E.E. degree from Manhattan College, the Bronx, in 1961 and the M.S. and Ph.D. degrees in electrical engineering from the University of Arizona, Tucson, in 1964 and 1966.

From February 1966 to September 1967 he worked in the Research Department of Sperry Flight Systems, Phoenix, Ariz. Since September 1967 he has been on the Faculty of Electrical Engineering, College of Engineering Sciences, Arizona State University, Tempe, where he is currently an Associate Professor, teaching graduate courses in control systems, computers, and random processes. He has been a consultant to Sperry Flight Systems on navigation, guidance, and control problems.

Dr. Higgins is a member of the Society for Computer Simulation and Eta Kappa Nu.