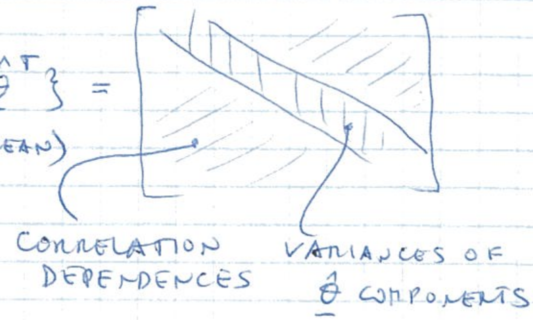
6. PROPERTIES OF ML ESTIMATOR

$$\underline{\underline{C}}_{\hat{\underline{\theta}}} = \mathrm{cov}\{\hat{\underline{\theta}}\} = \mathcal{E}\{\hat{\underline{\theta}}\,\hat{\underline{\theta}}^T\} =$$

(IF ZERO MEAN)



6.1 CRAMÉR-RAO LOWER BOUND

– LOWER BOUND ON COVARIANCE OF THE MODEL PARAMETERS INDEPENDENTLY OF THE ESTIMATOR (ALGORITHM)

(1)

LET $T = t(\underline{X})$ UNBIASED ESTIMATOR FOR $\psi(\theta)$ BASED ON DATA X

$$V = \frac{\partial}{\partial\theta} \ell(\theta) = \frac{\partial}{\partial\theta} \ln p(\underline{x},\theta) = \frac{1}{p}\frac{\partial}{\partial\theta} p \qquad p(\underline{x},\theta) = L(\theta)$$

$$\mathcal{E}\{V\} = \int p\left[\frac{1}{p}\frac{\partial}{\partial\theta}p\right]d\underline{x} \underset{(*)}{=} \frac{\partial}{\partial\theta}\int p(\underline{x},\theta)\,d\underline{x} = \phi$$

$$\mathrm{cov}\{V,T\} = \mathcal{E}\{VT\} = \mathcal{E}\left\{T\left[\frac{1}{p}\frac{\partial p}{\partial\theta}\right]\right\} = \int t(\underline{x})\left[\frac{1}{p}\frac{\partial p}{\partial\theta}\right]p(\underline{x})\,d\underline{x}$$

$$\underset{(*)}{=} \frac{\partial}{\partial\theta}\left[\int t(x)\,p(\underline{x},\theta)\,dx\right] = \psi'(\theta) \qquad \frac{\partial}{\partial\theta}\mathcal{E}\{T\}$$

$$\mathrm{var}(T)\,\mathrm{var}(V) \geq |\mathrm{cov}(V,T)|^2 = |\psi'(\theta)|^2 \qquad \text{(Schwarz-Cauchy)}$$

$$\mathrm{var}(V) = \mathcal{E}\left\{\left(\frac{\partial}{\partial\theta}\ln p(\underline{x},\theta)\right)^2\right\} = \mathbb{I}(\theta) \qquad \text{FISHER-INFORMATION (MATRIX)}$$

$$\left[\mathrm{var}(T) \geq \frac{|\psi'(\theta)|^2}{\mathrm{var}(V)} = \frac{|\psi'(\theta)|^2}{\mathbb{I}(\theta)}\right]$$

UNDER CONDITIONS (*)

- $\forall x,\ p(x,\theta) > \phi$
  $\frac{\partial}{\partial\theta}\ln p$ EXISTS, FINITE
- $\frac{\partial}{\partial\theta}\left[\int T p\,dx\right] = \int T(x)\left[\frac{\partial}{\partial\theta}p\right]dx$
- SUPPORT BOUNDS DO NOT DEPEND ON θ !
- IF INFINITE SUPPORT, THEN CONT. DIFFERENTIABLE

(2) $$\boxed{\mathbb{I}(\theta) = \mathcal{E}\left\{\left(\frac{\partial \ell}{\partial\theta}\right)^2\right\} = -\mathcal{E}\left\{\frac{\partial^2 \ell}{\partial\theta^2}\right\}}$$

$$\ell = \ln L(\theta) = \ln p(x,\theta)$$

(3) FOR $\psi(\theta) = \theta$

$$\left[\mathrm{var}(T) \geq \frac{1}{\mathbb{I}(\theta)}\right]$$

(4) FOR ESTIMATOR WITH GIVEN BIAS: $b(\theta) = \mathcal{E}\{\hat{\theta}\} - \theta$

$$\psi(\theta) = b(\theta) + \theta$$

$$\left[\mathrm{var}(T) \geq \frac{|1 + b'(\theta)|^2}{\mathbb{I}(\theta)}\right]$$