Measurement Theory Homework #1

This homework covers estimation theory. The homework is parametrized separately for each individual (see table on the third page). To solve the problems the use of MATLAB ([1], [2], [3], [4]) is recommended, but the use of any similar program is allowed.

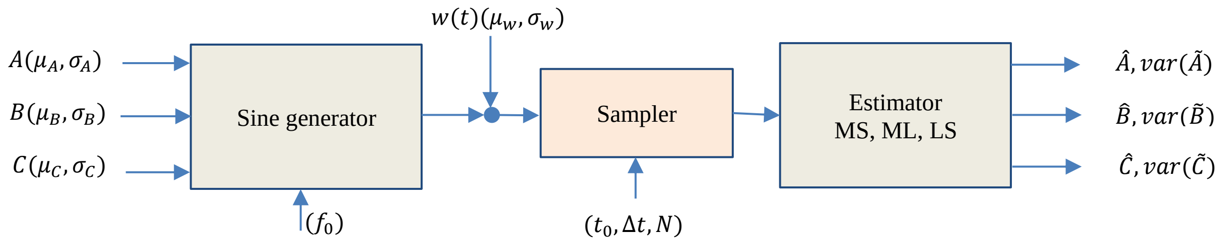
1. Implement please a signal generator producing signal having time function $u(t) = Asin(2\pi f_0 t) + Bcos(2\pi f_0 t) + C$! The value of the frequency is considered completely accurate, but A, B and C are uncertain. This uncertainty is expressed by the fact that these values are considered as Gaussian random variables with mean value μ_A, μ_B and μ_C, and covariance $\boldsymbol{C}_{aa}$.

$$\boldsymbol{C}_{aa} = \sigma_a^2 \begin{bmatrix} 1 & \rho_1 & \rho_1^2 \\ \rho_1 & 1 & \rho_1 \\ \rho_1^2 & \rho_1 & 1 \end{bmatrix},$$

Assuming a given representation of these random variables take samples from this signal at $t = t_0 + n\Delta t$ $(n = 0,1,\dots,N-1)$ knowing that the measurements are distorted by additive Gaussian noise, $w(t)$, having $N(\boldsymbol{0}, \boldsymbol{C})$!

$$\boldsymbol{C} = \sigma_w^2 \begin{bmatrix} 1 & \rho_2 & \rho_2^2 & \cdots & \rho_2^{N-1} \\ \rho_2 & 1 & \rho_2 & \cdots & \vdots \\ \rho_2^2 & \rho_2 & 1 & \cdots & \rho_2^2 \\ \vdots & \vdots & \vdots & \ddots & \rho_2 \\ \rho_2^{N-1} & \cdots & \rho_2^2 & \rho_2 & 1 \end{bmatrix}$$

I.e. the task is to start at $t = 0$ with a sine wave of frequency f_0 having random parameters including a DC component. To this a random noise (channel noise) is added. Having the noisy samples, parameter estimation is to be performed. The random parameters and the samples of the additive noise are correlated that is parametrized by ρ_1 and ρ_2, respectively (max 6 points).



At this point we have a simulation environment suitable for making experiences. Since in this case we generate the values to be measured, we will be able to test the efficiency of our methods.

2. Based on the available information choose an estimation method for getting the optimum estimate of parameters A, B and C, the covariance of the estimation error and the (conditional) bias[1]! Please give the expressions for the calculations! Perform the measurement based on $N = 5, 10, 100$ uniformly distributed signal samples[2] first in such a way that the samples are from a complete period of the signal, and then from one tenth of a period! Repeat the measurements for the case of $\rho_1 = \rho_2 = 0$! Please, fit the results into a table and compare them with the true values, and the parameters of the random processes! Evaluate the results! Do they correspond to your prior expectation? (max. 8 points)

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[1] The applied procedure fits a sine wave to the measured signal. Sine wave fitting is widely used from the investigation of energy distribution systems to the testing of A/D converters (ADCs). In the case of this latter we put a sine wave consisting of multiple periods to the input of the ADC and based on the digital output values we perform a so-called histogram test. Applying this test, we make statistics from the occurrence of the excitation signal samples in the digital domain, assuming that the excitation signal parameters are known. The signal transition levels of the ADC can be calculated from these statistics if the parameters of the sine wave are known. Since the sine wave is uncertain due to distortions and disturbances the step of fitting a sine wave is unavoidable.

[2] If we increase the number of signal samples only the effect of the additive noise can be reduced, concerning the unknown parameters further information will not be available.

3. Using the random signal and noise parameters generated in 1, also compute the Maximum Likelihood (ML) estimate and covariance, and the least-squares (LS) estimates of the unknown parameters for all the three N values and both sampling frequencies! Repeat the measurements for the case of $\rho_2 = 0$! Evaluate the results and compare them with those in the table summarizing the results of 2! Which estimate is the best? (max. 6 points)
4. Repeat the measurement of 2 *M*-times[3], and compute the mean of the estimates, furthermore the mean of the squared deviation of the estimates from the average! Evaluate the results and compare them with the a priori known properties and those in the table summarizing the results of 2 (max. 5 points)! Obviously here every measurement is based on different random values of the signal parameters and a new random noise sequence.
5. Similarly to 4, using the same random signal parameters and noise sequences, compute the mean of the LS estimates, and the mean of the squared deviation of the estimates from the average! Evaluate the results and compare them with the a priori known properties and with those in the table summarizing the results of 2 (max. 3 points)!
6. Imagine that the signal generator operates according to the parameter estimates of 2! Give the CRLB of the frequency f_0 as a function of this frequency! Give the result also in graphical form for all the three values of N! Set the frequency value and step within the range $[0, 1/(2\Delta t)]$ corresponding to the needs of the graphical representation! What is your experience? What kind of design considerations can be proposed? (max. 6 points)
7. Using the generated signal parameters, repeat the LS estimation of point 3 for such a case, where the additive distortion[4] has the form of $w'(t) = w(t) + au^q(t)$! What is your experience? What is your suggestion? (max. 6 points)

Consider the fact that you can compare the numerical results received in 2, 3 and 7 only if the very same record is processed, i.e., you should generate a random value for A, B, C and the $N = 100$ channel noise samples only once! (If less is needed then use as many as you need.) You must follow this strategy also in the case of 4 and 5, i.e., for both you should use the very same M records! (Remember, however, that here every record is to be generated with different random A, B, C and channel noise values since you are computing statistics!)

In addition to the above requirements, the commented program list and a short summary of the gained experiences are also requested to submit. Please also add your name, signature, Neptun-code, and email-address.

Submission is requested via the homework portal https://hf.mit.bme.hu as a single file in *pdf* format.

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The condition of acceptance: min. **16 points** (40%). **Good luck!**

[3] Repeating the experiments our expectation is that we get further information about the unknown parameters, and the averaging of the individual estimates will further reduce the effect of the additive noise.

[4] Here we have both deterministic and random distorting effects. We can try to compensate the deterministic component.

Neptun code	μ_A	μ_B	μ_C	σ_a	σ_w	ρ_1	ρ_2	t_0	f_0	*M*	*a*	*q*
	V	V	V	V	V			ms	Hz			
CI6GN0	.5	.5	.5	.1	.1	.1	.1	5	40	50	.1	2
CJJFTJ	1	1	1	.1	.1	.1	.1	10	50	60	.08	2
EF1IQD	1	-1	2	.1	.1	.2	.1	10	50	70	.1	3
F8S52E	1	2	1	.1	.2	.1	.2	10	50	50	.02	2
FGB002	2	1	1	.2	.3	.1	.3	5	50	60	.1	3
H3I99B	1	-2	2	.2	.2	.2	.2	5	50	70	.1	2
H8B414	2	2	1	.2	.1	.1	.1	5	50	80	.03	2
HY0C3E	2	1	-2	.3	.1	.2	.1	5	50	90	.1	3
ITGBNO	2	2	2	.1	.3	.1	.3	1	40	100	.01	2
J1QWH1	.5	-1	1	.2	.3	.1	.3	2	40	110	.07	3
LUC34P	1	.5	1	.3	.1	.2	.1	3	40	120	.2	2
M2JXJY	1	1	.5	.2	.3	.1	.3	4	40	110	.03	3
NAD7Y8	.5	.5	.5	.1	.1	.1	.1	5	40	100	.1	2
PLEUO1	2	2	1	.2	.1	.1	.1	5	40	90	.03	2
Q5ZGIR	2	-1	2	.3	.1	.2	.1	5	40	80	.1	3
RXKUBJ	1	2	1	.1	.2	.1	.1	5	40	50	.1	2
SUT0TD	2	1	1	.2	.3	.1	.1	10	50	60	.08	2
TRKRXD	1	-2	2	.2	.2	.2	.1	10	50	70	.1	3
U16MCS	2	2	1	.2	.1	.1	.2	10	50	50	.02	2