

Bias of Mean Value and Mean Square Value Measurements Based on Quantized Data

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Abstract— This paper investigates the imperfect fulfillment of the validity conditions of the noise model quantization. The general expressions of the deviations of the moments from Sheppard's corrections are derived. Approximate upper and lower bounds of the bias are given for the measurement of first- and second-order moments of sinusoidal, uniformly distributed, and Gaussian signals. It is shown that because of the uncontrollable mean value at the input of the ADC (offset, drift), the worst-case values have to be investigated; it is illustrated how a simple-form envelope function of the errors can be used as an upper bound. Since the worst-case relative positions of the signal and the quantization characteristics are taken into account, the results are valid for both midtread and midrise quantizers, while in the literature results are given for a selected quantizer type only.

I. INTRODUCTION

FROM the statistical point of view, uniform quantization can usually be described by the so-called noise model: the effect of the quantizer is modeled by an additive, independent noise, uniformly distributed in $(-q/2, q/2)$, where q is the quantum size [1]–[7]. Moreover, it is often noted that the spectrum of the quantization noise samples is white [4], [7]. Usually, it is mentioned that the noise model is applicable if the quantum size q is sufficiently small when compared to the amplitude or the standard deviation of the signal, and no saturation of the quantizer occurs.

When the noise model may be applied, the mean value of the quantized data will be exactly equal to the mean value of the nonquantized random variable.

$$\mathcal{E}\{x_q\} = \mathcal{E}\{x + n_q\} = \mathcal{E}\{x\} + \mathcal{E}\{n_q\} = \mathcal{E}\{x\} \quad (1)$$

since the mean value of n_q is zero because of its uniform distribution in $(-q/2, q/2)$. Similarly, for the mean square values, the following expression can be obtained:

$$\begin{aligned} \mathcal{E}\{x_q^2\} &= \mathcal{E}\{x^2\} + 2\mathcal{E}\{xn_q\} + \mathcal{E}\{n_q^2\} \\ &= \mathcal{E}\{x^2\} + \frac{q^2}{12}. \end{aligned} \quad (2)$$

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Equations (1) and (2) yield the so-called Sheppard's corrections [1]–[3] for the first-order and the second-order case, respectively,

$$\begin{aligned} \mathcal{E}\{x\} &= \mathcal{E}\{x_q\}, \\ \mathcal{E}\{x^2\} &= \mathcal{E}\{x_q^2\} - \frac{q^2}{12}. \end{aligned} \quad (3)$$

The quantization theorem [1]–[6] gives the applicability conditions of the noise model. The probability density function of the quantization error,¹ $n_q = x_q - x$, can be expressed in the form of a complex Fourier series [1]–[5]

$$\begin{aligned} f(z) &= \frac{1}{q} + \frac{1}{q} \sum_{k \neq 0} W_x \left(\frac{2\pi k}{q} \right) \exp \left(\frac{+j2\pi k z}{q} \right) \\ &\quad \text{for } -\frac{q}{2} \leq z \leq \frac{q}{2} \\ &= 0 \quad \text{otherwise.} \end{aligned} \quad (4)$$

This distribution will be uniform when the characteristic function of the random variable to be quantized

$$W(\alpha) = \int_{-\infty}^{\infty} f(x) e^{j\alpha x} dx \quad (5)$$

has zero values at certain points

$$\begin{aligned} W(\alpha) &= 0 \quad \text{for } \alpha = \frac{2\pi k}{q}, \\ &\quad k = \pm 1, \pm 2, \dots \end{aligned} \quad (6)$$

[4] and [5]. Condition (6), which is a sufficient and necessary one [4], holds e.g., when the characteristic function is bandlimited

$$W(\alpha) = 0 \quad \text{for } |\alpha| \geq \frac{2\pi}{q}. \quad (7)$$

This condition can be exactly fulfilled in theory only, since a bandlimited characteristic function implies nonbandlimitedness of the pdf (probability density function), which means the existence of infinitely large amplitudes. However, approximate fulfillment of the condition is possible.

Another family of distributions, with a uniform quantization error, consists of the distributions that can be decomposed into the convolution of two distributions, one of which is a uniformly distributed component in $(-rq/2, rq/2)$, with r being a positive integer. The convolution of two distributions

¹Note that some of the literature, e.g., [4] and [5], define the quantization error with the opposite sign, as $n_q = x - x_q$.

(or, in other words, of the two pdf's) corresponds to the summation of two independent random variables, and to the multiplication of the two characteristic functions. The characteristic function of this component is

$$\text{sinc}\left(\frac{rq\alpha}{2}\right) = \frac{\sin(rq\alpha/2)}{rq\alpha/2} \quad (8)$$

with zero values at the points $2\pi k/rq$, $k = \pm 1, \pm 2, \dots$. When such an independent random variable is added to another one, the resulting characteristic function also has zeros at the desired points.

When the quantization error has uniform distribution, Sheppard's second-order correction is valid if and only if $\mathcal{E}\{xn_q\} = 0$. The expression of the correlation was derived by Sripad and Snyder [4]

$$\mathcal{E}\{xn_q\} = \frac{q}{2\pi} \sum_{k \neq 0} \dot{W}_x \left(\frac{2\pi k}{q} \right) \frac{(-1)^{k+1}}{k} \quad (9)$$

where the overdot denotes derivation. The condition of the uncorrelatedness is, in addition to (6), that

$$\frac{dW(\alpha)}{d\alpha} = 0 \quad \text{for } \alpha = \frac{2\pi k}{q}, \quad k = \pm 1, \pm 2, \dots \quad (10)$$

A condition, similar to (7), is sufficient again

$$W(\alpha) = 0 \quad \text{for } |\alpha| \geq \frac{2\pi}{q} - \varepsilon, \quad \varepsilon > 0. \quad (11)$$

Another possibility for the fulfillment of (10) is that the random variable has an independent, random component with triangle distribution in $(-pq, pq)$ or, more generally, it has two independent, random components, uniformly distributed in $(-rq/2, rq/2)$ and $(-tq/2, tq/2)$, respectively, with p, r, t positive integers.

However, the above conditions of the validity of the noise model are usually not exactly met. In such cases, the question arises: how large is the deviation from the ideal case.

In this paper, this deviation is studied. In Section II, an example is discussed, and the basic ideas are developed. Section III presents the derivation of the general formulas, while Section IV applies these results to common signals: Gaussian, uniformly distributed, and sinusoidal ones.

II. AN EXAMPLE: QUANTIZATION OF A SINE WAVE

The arising phenomena can be studied with the following example. Fig. 1 shows two typical quantization error pdf's for sine waves² with peak value A and mean value μ . It is obvious that the deviation from the uniform distribution depends on the ratio A/q : for larger values, the uniform distribution will be better approximated in the sense that the integral of the absolute value of the difference between the error pdf and the uniform one will be small. It is also clear that the position of the interval of higher probability depends on the mean

²It should be mentioned here that, throughout this paper, it is assumed that the measured signal is stationary. This assumption is violated when, e.g., sampling is done synchronously with the sine wave; in such cases, the ideas of statistical quantization theory generally do not apply.

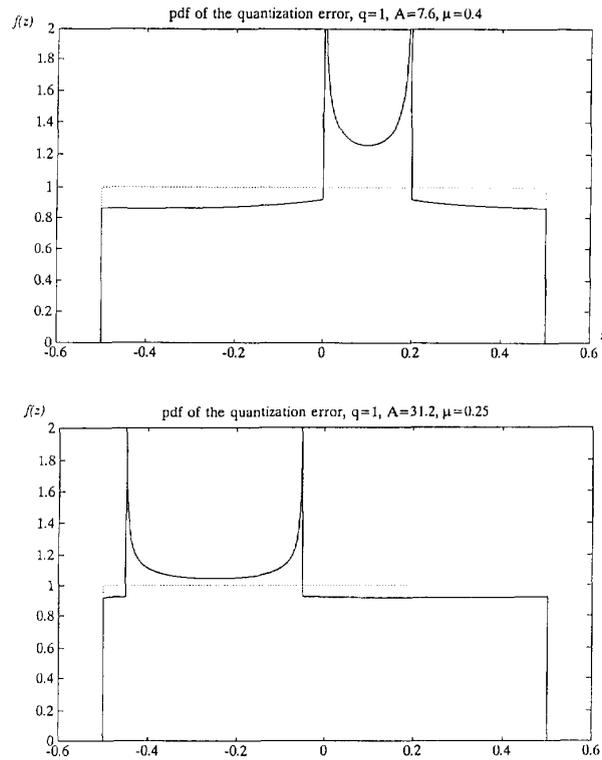


Fig. 1. Probability density functions of the quantization errors of sine waves for a rounding (midtread) quantizer.

value μ : by changing μ , the pdf will be circularly shifted in $(-q/2, q/2)$. When the higher probability interval is close to the left or to the right side, the mean value of the quantization error will significantly differ from zero; for the positions where the distribution of the quantization error is symmetrical, the mean is exactly zero. The second-order moment also depends on the mean value; therefore, it also has to be investigated as a function of μ .

In measurements, the mean value of the signal usually cannot be exactly controlled; moreover, this is often the quantity to be measured. Therefore, it is not known which value of μ occurred. There is no better way than to give an *upper limit* of the deviation from Sheppard's corrections. In the following sections, this limit will be given.

Fig. 2 illustrates the behavior of the mean value of the quantization error, as a function of μ , for $A = 7.6q$. Clearly, it seems to be hopeless to obtain a simple analytical form of this function. What can be done is to find the maximum in a numerical way, and to investigate the behavior of this maximum for all reasonable values of A . Since the pdf of the discrete random variable x_q can be easily calculated, the numerical computations do not require much computer power.

In special cases, also, the general summation formulas [see, e.g., (18) or (23)] can be evaluated. For example, [9] gives an easily calculable formula of the quantization error power for the case of a midrise quantizer (see below) and a zero-mean sine wave with amplitudes exactly equal to integer multiples of the quantum size. However, this evaluation cannot

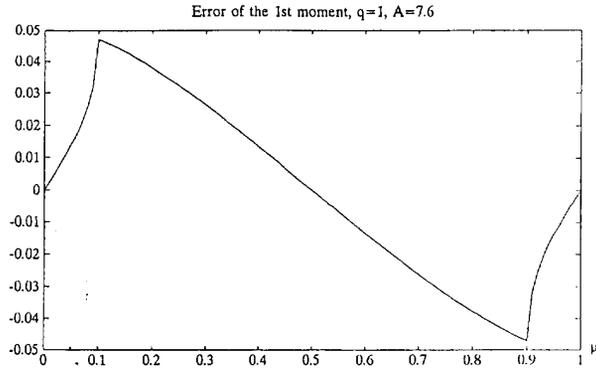


Fig. 2. The behavior of the mean value of the quantization error of a sine wave, as a function of μ .

be performed for other values of A and μ , while from Fig. 8 it can be seen that the largest deviations occur for noninteger values of A/q . Direct numerical calculations can be done for any case, in an effective way.

Fig. 3 shows the numerically calculated maximum mean values of the quantization error, as depending on the sine amplitude A . It is hopeless again to exactly determine the exact value of A in a practical situation, but the *envelope* of this function can be given in a simple form (see dotted line). It is our aim here to determine an appropriate expression for such envelopes of the first-order and second-order moments.

III. DERIVATION OF THE GENERAL EXPRESSIONS

In this paper, the expressions will be given for the case of a rounding (midtread) quantizer, that is, a uniform one with a dead zone around zero. It should be noted that, in some of the papers on quantization, the so-called midrise quantizer is treated (with a comparison level at zero).

A. First-Order Moment

The expected value of the noise can be easily calculated

$$\begin{aligned} \mathcal{E}\{n_q\} &= \int_{-\infty}^{\infty} z f_n(z) dz \\ &= \frac{q}{j2\pi} \sum_{k \neq 0} W_x \left(\frac{2\pi k}{q} \right) \frac{(-1)^k}{k}. \end{aligned} \quad (12)$$

This result can also be derived by applying

$$\mathcal{E}\{x_q\} = \frac{1}{j} = \left. \frac{dW_{x_q}(\alpha)}{d\alpha} \right|_{\alpha=0} \quad (13)$$

to the characteristic function of the quantized variable

$$\begin{aligned} W_{x_q}(\alpha) &= \sum_{k=-\infty}^{\infty} W_x \left(\alpha - \frac{2\pi k}{q} \right) \\ &\quad \times \text{sinc} \left(\frac{q\alpha}{2} - k\pi \right) \end{aligned} \quad (14)$$

with

$$\text{sinc}(x) = \frac{\sin(x)}{x}. \quad (15)$$

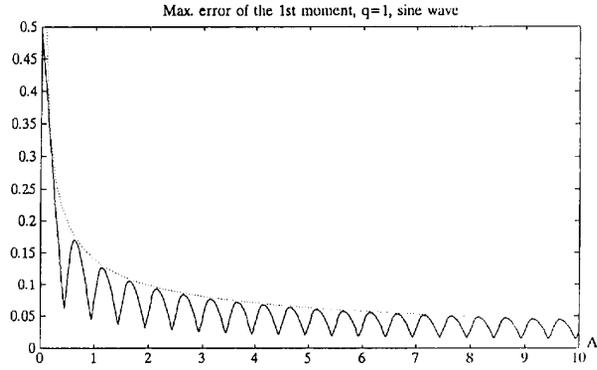


Fig. 3. The maximum values of the bias of the quantization error of a sine wave, depending on the signal amplitude A . The dotted line is the asymptotic envelope: $0.14/\sqrt{A}$.

Equation (12) can be transformed to a form where its dependence on the mean value μ is explicitly shown. By introducing the characteristic function of the zero mean random variable $x_0 = x - \mu$,

$$\begin{aligned} W_{x_0}(\alpha) &= \int_{-\infty}^{\infty} f_x(z + \mu) e^{j\alpha z} dz \\ &= e^{-j\alpha\mu} W_x(\alpha) \end{aligned} \quad (16)$$

$$\mathcal{E}\{n_q\} = \frac{q}{j2\pi} \sum_{k \neq 0} e^{j(2\pi k/q)\mu} W_{x_0} \left(\frac{2\pi k}{q} \right) \frac{(-1)^k}{k}. \quad (17)$$

This expression strongly depends on μ , and its worst value has to be considered. Because of the symmetry to zero (see Fig. 2), it is enough to consider the maximum value: $\max_{\mu}(\mathcal{E}\{n_q\})$.

For signal distributions, symmetric to their mean value, $W_{x_0}(2\pi k/q)$ is real and symmetric. In such a case,

$$\mathcal{E}\{n_q\} = \frac{q}{\pi} \sum_{k=1}^{\infty} \sin \left(\frac{2\pi k}{q} \mu \right) W_{x_0} \left(\frac{2\pi k}{q} \right) \frac{(-1)^k}{k}. \quad (18)$$

This is the Fourier series form of the mean value of the quantization error, as a function of μ . Its maximum value cannot be given in general. However, it is easy to see that the lower and upper limits of the maximum values of this function can be given by the effective value and the sum of the absolute values of the coefficients, respectively,

$$\begin{aligned} &\frac{q}{\pi} \sqrt{\sum_{k=1}^{\infty} \frac{1}{2} \cdot W_{x_0}^2 \left(\frac{2\pi k}{q} \right) \frac{1}{k^2}} \\ &\leq \max_{\mu}(\mathcal{E}\{n_q\}) \\ &\leq \frac{q}{\pi} \sum_{k=1}^{\infty} \left(\left| W_{x_0} \left(\frac{2\pi k}{q} \right) \right| \cdot \frac{1}{k} \right). \end{aligned} \quad (19)$$

In the case of a sine wave, for $A = 7.6q$, these bounds yield $0.027q \leq \max_{\mu}(\mathcal{E}n_q) \leq 0.075q$, in good agreement with Fig. 2, where $\max_{\mu}(\mathcal{E}n_q) \sim 0.047q$. However, these bounds can be somewhat pessimistic; the upper bound is 1.5 times larger than the true value.

From (19) it may be concluded that by increasing the amplitude of the random variable to be quantized, that is,

by contracting $W_{x0}(\alpha)$, the maximal bias will tend to zero similarly to the decrease of $W_{x0}(\alpha)$ [10]. A simple proof was given by Nagy [11] for the case when the decay of the characteristic function can be majored as follows:

$$|W_x(\alpha)| \leq \frac{c}{|D\alpha|^p}, \quad p > 0 \quad (20)$$

with D being the amplitude or the standard deviation of x . One obtains, by using the right side of (19),

$$\begin{aligned} |\mathcal{E}\{n_q\}| &\leq \frac{q}{\pi} \sum_{k=1}^{\infty} \frac{c}{\left(D \frac{2\pi k}{q}\right)^p} \cdot \frac{1}{k} \\ &= \frac{c \cdot 2 \left(\frac{q}{2\pi}\right)^{p+1} \sum_{k=1}^{\infty} \frac{1}{k^{p+1}}}{D^p} = d \left(\frac{q}{D}\right)^p \quad (21) \end{aligned}$$

with

$$\begin{aligned} d &= c \cdot 2 \left(\frac{1}{2\pi}\right)^{p+1} \sum_{k=1}^{\infty} \frac{1}{k^{p+1}} \\ &= c \cdot 2 \left(\frac{1}{2\pi}\right)^{p+1} \zeta(p+1) \end{aligned}$$

and ζ being the Riemann zeta function [12]. The values of ζ are for a few important cases: $\zeta(1.5) \approx 2.612$, $\zeta(2) = \pi^2/6 \approx 1.645$, $\zeta(2.5) \approx 1.342$, $\zeta(3) \approx 1.202$, $\zeta(4) = \pi^4/90 \approx 1.082$. This means that for $p > 3$, $\zeta(p+1) \approx 1$ is a reasonable approximation.

According to (21), the bias of the first moment is majored by a function, similarly decaying as $|W_x(\alpha)|$, if this can be described by (20). Therefore, in order to obtain a reasonable form of the envelope for large signal amplitudes, the asymptotic behavior of $W_{x0}(\alpha)$ has to be investigated. In many cases, however, the upper bound given by (21) is too pessimistic. For a sine wave, $c = \sqrt{2/\pi}$, $D = A$, and $p = 0.5$ [see (42) and (43)], which yields $d = 1/\pi^2 \zeta(1.5) \approx 0.265$, an upper limit quite larger than the best one 0.14, found in Fig. 3.

Before applying the above idea to common signal forms, let us investigate the approximation of Sheppard's second-order correction, too.

B. Second-Order Moment

The error of Sheppard's second-order correction may be manifested in two facts: $2\mathcal{E}\{xn_q\}$ is not zero, and/or $2\mathcal{E}\{n_q^2\}$ is not equal to $q^2/12$ [see also (2)]

$$\begin{aligned} e_2 &= \mathcal{E}\{x_q^2\} - \mathcal{E}\{x^2\} - \frac{q^2}{12} \\ &= 2\mathcal{E}\{xn_q\} + \mathcal{E}\{n_q^2\} - \frac{q^2}{12}. \quad (22) \end{aligned}$$

The correlation term is given by (9), and the mean square value of the noise is [4]

$$\mathcal{E}\{n_q^2\} = \frac{q^2}{12} + \frac{q^2}{2\pi^2} \sum_{k \neq 0} W_x\left(\frac{2\pi k}{q}\right) \frac{(-1)^k}{k^2}. \quad (23)$$

Using (16) again, and

$$\begin{aligned} \dot{W}_x(\alpha) &= e^{j\alpha\mu} \dot{W}_{x0}(\alpha) + j\mu e^{j\alpha\mu} W_{x0}(\alpha) \\ &= e^{j\alpha\mu} \left(\dot{W}_{x0}(\alpha) + j\mu W_{x0}(\alpha) \right) \quad (24) \end{aligned}$$

for symmetrical signal distributions, the following simpler expressions can be obtained:

$$\begin{aligned} \mathcal{E}\{xn_q\} &= \frac{q}{\pi} \sum_{k=1}^{\infty} \cos\left(\frac{2\pi k}{q}\mu\right) \dot{W}_{x0}\left(\frac{2\pi k}{q}\right) \frac{(-1)^{k+1}}{k} \\ &\quad + \frac{q}{\pi} \sum_{k=1}^{\infty} \sin\left(\frac{2\pi k}{q}\mu\right) \mu W_{x0}\left(\frac{2\pi k}{q}\right) \frac{(-1)^k}{k} \quad (25) \end{aligned}$$

$$\begin{aligned} \mathcal{E}\{n_q^2\} &= \frac{q^2}{12} + \frac{q^2}{\pi^2} \sum_{k \neq 0} \cos\left(\frac{2\pi k}{q}\mu\right) \\ &\quad \cdot W_{x0}\left(\frac{2\pi k}{q}\right) \frac{(-1)^k}{k^2}. \quad (26) \end{aligned}$$

The first term in (25) is periodic as a function of μ , and represents the covariance of x and n_q , while the second one is nothing else than $\mu\mathcal{E}\{n_q\}$ [see (18)]. Thus, (25) exactly represents the following expression for symmetrical signal distributions:

$$\mathcal{E}\{xn_q\} = \text{cov}\{x, n_q\} + \mu\mathcal{E}\{n_q\}. \quad (27)$$

From (18) and (26) or (12) and (23), the general expression of the variance of the quantization error can also be derived, using

$$\text{var}\{n_q\} = \mathcal{E}\{n_q^2\} - \mathcal{E}^2\{n_q\}. \quad (28)$$

It is obvious again that the deviation from Sheppard's second-order correction will tend to zero only if

- 1) μ is limited, and
- 2) the correlation coefficient between x and n_q

$$r_{xn} = \frac{\text{cov}\{x, n_q\}}{\sqrt{\text{var}\{x\} \cdot \text{var}\{n_q\}}} \quad (29)$$

disappears more quickly with an increasing signal amplitude than the increasing amplitude itself or the standard deviation of x . This is not the case when the derivatives of the contracting characteristic function do not disappear with an increasing amplitude of x . Since the contraction introduces a multiplier X

$$\frac{dW(X\alpha)}{d\alpha} = X\dot{W}(X\alpha) \quad (30)$$

the derivative of the characteristic function should tend to zero more quickly than $1/X$. Furthermore, it may be noticed that since $X\dot{W}(X\alpha)$ usually disappears more slowly with an increasing X than $W(X\alpha)$, the derivative will dominate in the asymptotic behavior of the envelope of the bias of the second-order moments.

If $\lim_{X \rightarrow \infty} \text{cov}\{xn_q\} = 0$ is not true, it may be reasonable to study the deviation after normalization by $\text{var}\{x^2\}$, as it will be done in Section IV, for the case of the sine wave.

For the second-order moments, there will be no symmetry, similar to first-order moments (see Fig. 2); therefore, we are going to examine both the maxima and the minima of the deviations.

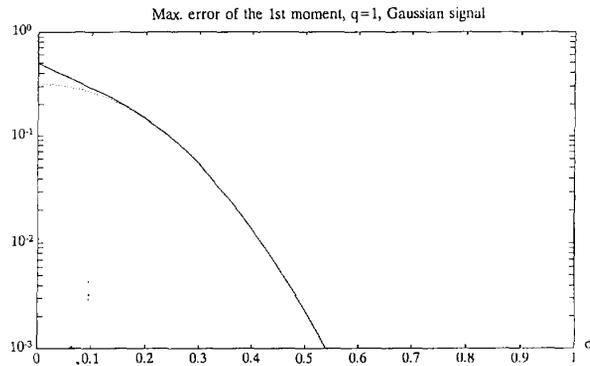


Fig. 4. The maximum values of the bias of the quantization error of a Gaussian noise, depending on the standard deviation σ . The dotted line is the asymptotic envelope, see (33).

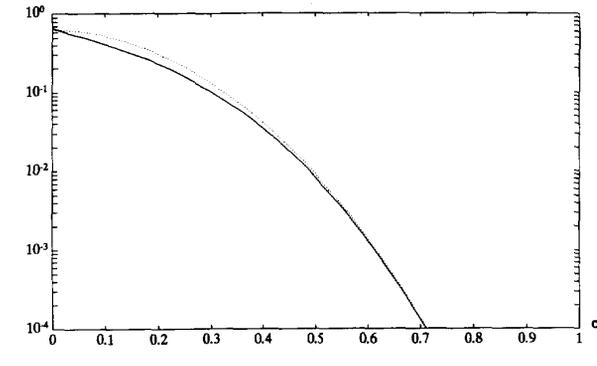


Fig. 5. The maximum values of the deviation from Sheppard's second-order correction for a Gaussian noise, depending on the standard deviation σ , for $0 \leq \mu \leq q$. The dotted line is the asymptotic envelope, see (35). Max. error of the 2nd moment, $q = 1$, Gaussian signal.

IV. APPLICATION TO SPECIAL SIGNAL FORMS

The above-discussed errors can be rather easily investigated by numerical methods in the case of simple distributions like Gaussian, uniform, and sinusoidal ones. We shall look for envelopes in a parametric form, determined on the basis of the above considerations.

A. Gaussian Noise

In the case of a Gaussian random variable, the characteristic function is

$$W_x(\alpha) = e^{j\alpha\mu} e^{-(\sigma^2\alpha^2/2)}. \tag{31}$$

This function is nowhere zero, but it very quickly disappears for $|\alpha| > 2\pi/\sigma$

$$\begin{aligned} W_{x0}\left(\frac{4\pi}{\sigma}\right) &= e^{-8\pi^2} < 10^{-25} \cdot e^{-2\pi^2} \\ &= 10^{-25} \cdot W_{x0}\left(\frac{2\pi}{\sigma}\right). \end{aligned} \tag{32}$$

Therefore, in the above infinite sums, the first terms dominate if $\sigma > q$

$$\max_{\mu} \{|\mathcal{E}\{n_q\}|\} \approx \frac{q}{\pi} e^{-(2\pi^2\sigma^2/q^2)} \tag{33}$$

and, using

$$W_{x0}(\alpha) = \sigma^2 \alpha e^{-(\sigma^2\alpha^2/2)} \tag{34}$$

and combining the amplitudes of the cos and sin terms,

$$\begin{aligned} \max_{\mu} \{e_2\} &= \max_{\mu} \left\{ \mathcal{E}\{x_q^2\} - \mathcal{E}\{x^2\} - \frac{q^2}{12} \right\} \\ &= \max_{\mu} \left\{ 2\mathcal{E}\{xn_q\} + \mathcal{E}\{n_q^2\} - \frac{q^2}{12} \right\} \\ &\approx \sqrt{\left(4\sigma^2 + \frac{q^2}{\pi^2}\right)^2 + \left(\frac{2q}{\pi}|\mu|_{\max}\right)^2} e^{-(2\pi^2\sigma^2/q^2)}. \end{aligned} \tag{35}$$

The minima of e_2 need not be investigated, since the good approximation by the first term of the series ensures the symmetry; thus, $\min_{\mu} \{e_2\} \approx -\max_{\mu} \{e_2\}$.

Both functions, defined by (33) and (35), disappear with an increasing value of σ , similar to W_{x0} and $W_{x0}(\alpha)$, respectively (Figs. 4 and 5).

B. Uniformly Distributed Noise

The characteristic function disappears much more slowly than for the Gaussian signal

$$W_x(\alpha) = e^{j\alpha\mu} \frac{\sin(A\alpha)}{A\alpha} \tag{36}$$

where the uniform distribution is in the interval $(-A + \mu, A + \mu)$. It is easy to see that (6) is fulfilled for $A = rq/2$, $r = 1, 2, \dots$; therefore, (12) is exactly zero for these values, and may be different from zero elsewhere. $\mathcal{E}\{n_q\}$ can be easily evaluated, since $f_{n_q}(z)$ has a special step-wise form, and the bias is the largest when the higher level is just at one side. For $A = rq/2 + \Delta A$ with $\Delta A < q/2$, this is achieved for $\mu = q/2 - \Delta A$. The constant part gives zero, while the rest is the error

$$\max_{\mu} \{\mathcal{E}\{n_q\}\} = \frac{1}{2A} \int_{q/2-2\Delta A}^{q/2} z dz = \frac{q2\Delta A - (2\Delta A)^2}{4A}. \tag{37}$$

According to the above considerations, this function will have an envelope of the form c/A . It is easy to see that the maximum of the numerator is $q^2/4$; therefore, the envelope is (see Fig. 6)

$$\text{env}(A) = \frac{q^2}{16A}. \tag{38}$$

Indeed, this envelope fits the error function. The majoring function, given by (21), is somewhat larger: $\text{env}(A) \leq 2(1/2\pi)^2 \cdot \pi^2/6(q/A)q = (q/12A)q$, an indication of the fact that the upper bound given by (21) is somewhat pessimistic. The cause is that on the uniform grid of $2\pi k/q$, we cannot select such points only of $W_{x0}(\alpha)$, which are all on its positive envelope.

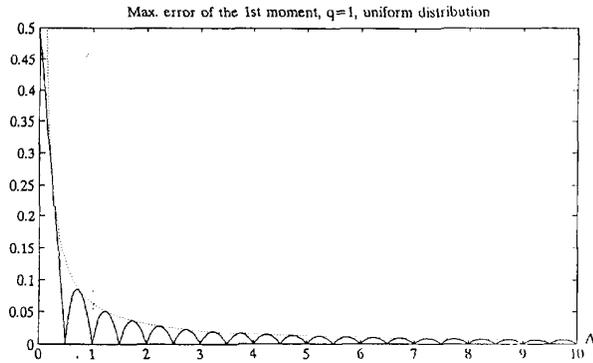


Fig. 6. The maximum values of the bias of the quantization error of a uniform noise in $(-A, A)$, depending on the signal amplitude A . The dotted line is the asymptotic envelope: $1/(16A)$.

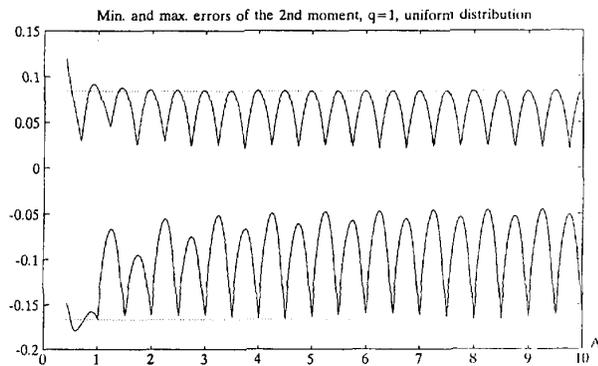


Fig. 7. The worst-case values of the deviation from Sheppard's second-order correction for a uniform noise in $(-A, A)$, depending on the signal amplitude A , for $0 \leq \mu \leq q$. The dotted lines are the asymptotic envelopes: $1/12$ and $-1/6$.

It may also be noticed that $\dot{W}_{x0}(2\pi k/q)$ will not disappear with an increasing A [see the first term of the right side of (39)]

$$\dot{W}_{x0}\left(\frac{2\pi k}{q}\right) = \frac{\cos\left(\frac{2\pi k}{q}A\right)}{\frac{2\pi k}{q}} - \frac{\sin\left(\frac{2\pi k}{q}A\right)}{\left(\frac{2\pi k}{q}\right)^2 A} \quad (39)$$

Therefore, the minimum and maximum errors of Sheppard's second-order correction will have finite value for $A \rightarrow \infty$ (see Fig. 7). From (22), (25), (26), and (39), these values can be obtained, using the dominance of the derivatives of the characteristic function and of the first term of (39), for large values of A

$$\begin{aligned} e_2 &\approx 2\mathcal{E}\{xn_q\} \\ &= 2 \cdot \frac{q}{\pi} \sum_{k=1}^{\infty} \cos\left(\frac{2\pi k}{q}\mu\right) \frac{\cos\left(\frac{2\pi k}{q}A\right)}{\frac{2\pi k}{q}} \cdot \frac{(-1)^{k+1}}{k} \\ &= \frac{q^2}{\pi^2} \sum_{k=1}^{\infty} \cos\left(\frac{2\pi k}{q}\mu\right) \cos\left(\frac{2\pi k}{q}A\right) \frac{(-1)^{k+1}}{k^2} \quad (40) \end{aligned}$$

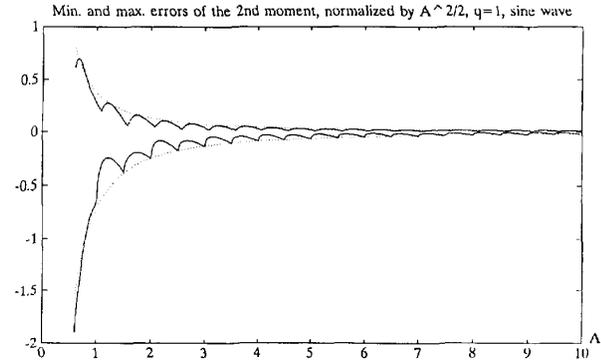


Fig. 8. The worst-case values of the deviation from Sheppard's second-order correction for a sine wave, normalized by $\text{var}\{x\} = A^2/2$ versus the signal amplitude A , for $0 \leq \mu \leq q$. The dotted lines are the asymptotic envelopes: $0.36/\sqrt{A^3}$ and $-0.72/\sqrt{A^3}$.

It may be speculated that the extreme values will be taken when the cosines have values ± 1 . This may result in terms 1, 1, 1, 1, \dots , or $-1, 1, -1, 1, -1, 1, \dots$, for $k = 1, 2, 3, 4, 5, \dots$, in the case of $A = rq$ or $A = (r + 0.5)q$, for $\mu = 0$ or $\mu = 0.5q$. These two cases yield (see Fig. 7)

$$\begin{aligned} \max\{e_2\} &= \frac{q^2}{\pi^2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = \frac{q^2}{12}, \\ \min\{e_2\} &= \frac{q^2}{\pi^2} \sum_{k=1}^{\infty} \frac{-1}{k^2} = -\frac{q^2}{6}. \quad (41) \end{aligned}$$

Indeed, for $A = 0.5rq$, these extreme values were obtained by the numerical calculations (Fig. 7).

C. Sine Wave

The characteristic function is

$$W_x(\alpha) = e^{j\alpha\mu} J_0(A\alpha) \quad (42)$$

where $J_0(z)$ is the zero-order Bessel function of the first kind. It is not easy to evaluate the above sums with the Bessel terms. However, it may be noticed that for large values of z ($z \gg 1$), $J_0(z)$ may be approximated [13] by

$$J_0(z) \approx \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi}{4}\right). \quad (43)$$

This means that for $A \gg q$, the characteristic function has a decay of $1/\sqrt{\alpha}$ character; therefore, the envelope of the error of Sheppard's first-order correction should also behave similarly. Indeed, such a function with a proper constant ($\text{env}(A) \approx 0.14\sqrt{q/A} \cdot q$) can seemingly be used as an envelope (Fig. 3). As we saw earlier, the upper limit, calculated from (21), is again somewhat larger: $\text{env}(A) \leq 0.265\sqrt{q/A} \cdot q$.

As was mentioned above, the envelopes of the error of Sheppard's second-order correction have a $c\sqrt{A}$ character; that is why for a plot it is reasonable to normalize the error by $\text{var}\{x\} = A^2/2$. When normalized, it has envelopes of the form $0.36q^2/\sqrt{A/q^3}$ and $-0.72q^2/\sqrt{A/q^3}$ (Fig. 8).

V. CONCLUSIONS

The general formulas of the deviations from Sheppard's corrections have been developed for the case of a uniform, midtread quantizer, provided that no saturation occurs. It has been shown that the deviations strongly depend on the mean value of the signal. The errors can be characterized by their asymptotic envelopes for $A \gg q$, and that these envelopes decay in a similar way as the characteristic functions do. These results, since they treat the worst-case mean values of the input signal, equally apply for the case of midrise quantizers, and can also be used for arbitrary waveforms, for which the pdf and the characteristic function can be given.

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