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# A Fast Sine-Wave Fit Algorithm

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**Abstract:** In testing digital waveform recorders, an important part is to fit a sinusoidal model to recorded data, and calculate the parameters that result in the best fit. Methods are already standardized; however, they demand high computational power. In this article a new, quick and accurate sine-fitting algorithm will be shown based on Levenberg-Marquardt (LM) method. The constraints of convergence will be discussed and an additional algorithm will be shown that is able to fulfil the necessary constraints and correctly initialize the iterative part even in the case of noisy and unknown sinusoid inputs.

**Key words:** AD converter test, sinusoidal fitting, four-parameter optimization

## 1 Introduction

Several measurements in testing of digital waveform recorders and AD converters or DSP algorithms are based on fitting a sinusoid waveform to the recorded samples in least squares sense. For this, two sinusoidal models can be used:

$$x(t) = A\sin(\omega t) + B\cos(\omega t) + C\,, \tag{1}$$

or

$$x(t) = \widetilde{A} \cdot \cos(\omega t + \Phi) + C. \tag{2}$$

If the frequency of the sinusoid is exactly known, the task is easy. By using eq. (1), the other three parameters of the sinusoid can be expressed by the direct solution of an equation system. This task is simply called three-parameter optimization. The detailed solution method can be found e.g. in [1-3]. In this case eq. (2) is not recommended, because there the parameters are not linear and direct solution is not possible.

Even if the nominal frequency of the sampler and the input sinusoid is given, the real frequencies can be slightly different. It can make huge differences in the measured AD parameters, therefore, the sinusoid frequency is usually assumed to be unknown and also has to be determined. This is called four-parameter optimization. In this case the estimation is nonlinear with respect to the parameters and the solution can be found only with iterative methods.

The possible iterative methods already can be found in the IEEE 1057 and 1241 standards, but several details are not told there (because this is not a requirement from the standard). Realizations can be found e.g. in [3-6]. These realizations differ in the search of initial estimates for the iterative parts. One of them works with grid method [3], the other works with Fourier interpolation [6]. These are very special solutions and calculation times of methods are not analyzed in details. Beside this, convergence and stability of the iterative part is also not well analyzed. In the followings, we will analyze the four-parameter optimization both for calculation time and for convergence speed. In ch 2, we will check the convergence. In ch 3 we will compare the speed of the methods based on the two sinusoidal models. In ch 4 we will check the possibilities to improve numerical stability. In ch 5 we will analyze the convergence limits and we will give a method to determine a correct initial phase value. Ch 6 gives the conclusions.

## 2 Convergence

An iterative method on the four-parameter optimization has to minimize the following cost function:

$$C=\int_0^T (A\cos(\omega t)+B\sin(\omega t)+C - \hat{A}\cos(\hat{\omega} t)-\hat{B}\sin(\omega t)-\hat{C})^2\,dt \quad (3)$$

or

$$C=\int_0^T \left(\tilde{A}\cos(\omega t+\Phi)+C-\hat{\tilde{A}}\cos(\hat{\omega} t+\hat{\Phi})-\hat{C}\right)^2 dt \quad (4)$$

where $[0,T]$ is the observation interval and operators with hat are the actual estimates. These integral can be expressed analytically (we don't show this formula here, because it is so long), hence convergence analysis can be easily done.

The analysis shows that the convergence of the model in eq. (1) is insensitive for the starting value of $A$, $B$ and $C$. The accuracy of the initial frequency must be <0.5 $f_s$/N, where N is the number of samples.

The model in eq. (2) is insensitive for the starting value of $\tilde{A}$ and $C$, however it is sensitive for the starting phase and frequency. The convergence limit of starting phase depends on the accuracy of the frequency estimate and vica verse:

Table 1. Convergence phase limits as a function of initial frequency deviation

| $\pm\Delta f\cdot N/f_s$ | $\pm\Delta\Phi$ [deg] |
|---|---|
| 0.66 | 0 |
| 0.5 | 60 |
| 0.33 | 100 |
| 0.17 | 135 |
| 0 | 180 |

## 3 Speed of methods

The four parameter optimization proposed by the standard is Newton-Gauss iteration [3]. If the parameter vector is

$$\underline{p}=[A,B,C,\omega] \text{ or } \underline{p}=[\tilde{A},\Phi,C,\omega], \quad (5)$$

and the measurement data of the sinusoid is:

$$\underline{x}^T=[x_1,\ldots,x_N] \quad (6)$$

then, after determining an initial parameter set, a modification vector of the parameters can be expressed as

$$\begin{aligned}\delta\underline{p}&=(\underline{\underline{J}}^T\underline{\underline{J}})^{-1}\underline{\underline{J}}^T\underline{x}\\ \underline{p}_{i+1}&=\underline{p}_i+\delta\underline{p}\end{aligned} \quad (7)$$

where $\underline{\underline{J}}$ is the Jacobian matrix,

$$\begin{pmatrix}\cos(\omega t_1) & \sin(\omega t_1) & 1 & -At_1\sin(\omega t_1)+Bt_1\cos(\omega t_1)\\ \vdots & \vdots & \vdots & \vdots\\ \cos(\omega t_N) & \sin(\omega t_N) & 1 & -At_N\sin(\omega t_N)+Bt_N\cos(\omega t_N)\end{pmatrix} \quad (8)$$

for the model in eq. (1), and

$$\begin{pmatrix}\cos(\omega t_1+\Phi) & -\tilde{A}t_1\sin(\omega t_1+\Phi) & 1 & -\tilde{A}\sin(\omega t_1+\Phi)\\ \vdots & \vdots & \vdots & \vdots\\ \cos(\omega t_N+\Phi) & -\tilde{A}t_1\sin(\omega t_N+\Phi) & 1 & -\tilde{A}\sin(\omega t_N+\Phi)\end{pmatrix} \quad (9)$$

for the model in eq. (2).

The calculation time of $\underline{\underline{J}}^T\underline{\underline{J}}$ is different at the two models. The result is a 4-by-4 matrix in both cases, which contains N-long sum of sine and cosine multiplicatives. The number of independent cross-multiplicatives by using Jacobian matrix from eq. (8) is 12:

$$\begin{aligned}&\sum\sin(k), \sum\cos(k), \sum\sin^2(k), \sum\cos^2(k),\\ &\sum t\sin(k), \sum t\cos(k), \sum t\sin^2(k), \sum t\cos^2(k),\\ &\sum t^2\sin^2(k), \sum t^2\cos^2(k), \sum\sin(k)\cos(k),\\ &\sum k\sin(k)\cos(k)\end{aligned} \quad (10)$$

By using eq. (9), the number of independent cross-multiplicatives is only 9:

$$\begin{aligned}&\sum\sin(k+\Phi), \sum\cos(k+\Phi), \sum\sin^2(k+\Phi),\\ &\sum\cos^2(k+\Phi), \sum k\cos(k+\Phi), \sum k\cos^2(k+\Phi),\\ &\sum k^2\cos^2(k+\Phi), \sum\sin(k+\Phi)\cos(k+\Phi),\\ &\sum k\sin(k+\Phi)\cos(k+\Phi),\end{aligned} \quad (11)$$

Hence the calculation by eq. (2) will be about 25% faster at one iteration step. This speed gain is worthwhile to choose the second model, if the number of iterations is the same at the two models. Simulations show, that the required number of iterations is the same, if the initial parameters are “close” to the real ones, that meant in the simulations: phase difference is <45 deg, amplitude error <10%, DC offset error <amplitude, frequency error <0.25 $f_s$/N.

## 4 Stability

Eq. (7) contains matrix inversion that can be numerically unstable. One possibility is, when one or more eigenvalues of the matrix to be inverted is 0. This can happen at an extreme initial parameter (e.g. 0 amplitude or frequency) that can be avoided by proper programming.

The other source of instability is the fixed length number representation that can cause problems, when the condition number of the matrix (ratio of the smallest and highest eigenvalues) is very high. Simulations show that a $10^{16}$ order of condition value is usual. It can cause problems at single precision calculations.

Another problem is that both fitting models are non-linear at least in one parameter. Newton-Gauss iteration works well only in linear equations. In cases, when the starting point is far from the optimum, the method can become divergent [7].

Numerical problems of inversion can be eliminated by adding a unity matrix to the original, multiplied by a small number, $\lambda$. In this case, all eigenvalues of the matrix will be $\geq \lambda$. Hence, eq. (7) will be modified to

$$\delta p = (\underline{\underline{J}}^T \underline{\underline{J}} + \lambda \underline{\underline{E}})^{-1} \underline{\underline{J}}^T \underline{x} . \qquad (12)$$

A good estimate for $\lambda$ is the trace of the matrix multiplied by the numerical accuracy of calculations [7] (e.g. $10^{-7}$ at single precision floating point).

This method also helps to solve divergence problems caused by non-linearity. The method is a trade-off between two iterative methods: by choosing $\lambda$ to 0, we will get back the original Newton-Gauss method. Taking out the quadratic matrix and keeping only the small $\lambda$, we will get to the gradient method that is more stable.

This $\lambda$-method can be used already, without any change. If we go further, we can use a more sophisticated algorithm called Levenberg-Marquardt method, where $\lambda$ is varied in the function of convergence with a very simple algorithm. A good description of the method can be found e.g. in [8]. In our simulations we found out that L-M method is not necessary, because the $\lambda$-method with our $\lambda$ estimate is very robust and quite close to N-G method that has the highest convergence speed. Run-time of L-M method is a few percent longer.

## 5 Initial parameters

### 5.1 DC offset

Analysis of eq. (3) and (4) and also simulations show that the DC offset initial parameter does not influence convergence too much and simply the average of samples can be used to estimate it. This method gives always a better estimate than the amplitude of the sinusoid. This estimate already acceptable and gives the same iteration number for model 1 and 2.

### 5.2 Amplitude, frequency and phase

Simulations show that the model in eq. (2) is very insensitive for the starting value of amplitude. The model in eq. (1) is also almost insensitive for the starting parameter of A and B and the iterations are always convergent. These parameters can be derived from amplitude and phase as

$$A = \widetilde{A} \cdot \cos(\Phi), \quad B = \widetilde{A} \cdot \sin(\Phi) \qquad (13)$$

So the proper value of A and B requires the knowledge of amplitude and phase information as well.

Both models are very sensitive for the value of the starting frequency and the second model is sensitive for phase value as well, therefore we have to use efficient, simple and non-iterative frequency and phase estimators to make the methods convergent and make the iteration of the second method as fast as the first one.

The solution can be found in [9], which is based on the 3 highest frequency bin of the Fourier-transform of the signal:

$$c_k = \frac{\exp(2\pi j \delta) - 1}{4\pi j(\delta - k)}$$

$$\widetilde{A} = \frac{1}{N} \cdot \frac{\sum_{k=-1}^{1} \left| Y_{j+k} \cdot \overline{c}_k \right|}{\sum_{k=-1}^{1} \left| c_k \right|^2} \qquad (14)$$

$$\Phi = \arg\left( \sum_{k=-1}^{1} Y_{j+k} \cdot \overline{c}_k \right)$$

where $Y_j$ is the highest amplitude complex frequency bin and $N$ is the number of samples used. $\delta$ can be calculated as

$$\alpha_1 = \mathrm{Re}\left\{\frac{Y_{j-1}}{Y_j}\right\}, \alpha_2 = \mathrm{Re}\left\{\frac{Y_{j+1}}{Y_j}\right\}$$
$$\delta_1 = \frac{\alpha_1}{1-\alpha_1}, \delta_2 = \frac{\alpha_2}{1-\alpha_2} \quad (15)$$
$$if\,(\delta_1 > 0)\,\&\,(\delta_2 > 0)\delta = \delta_2$$
$$else\,\delta = \delta_1$$

(Another accurate method also can be used for determining $\delta$, depicted in [10], but this is not examined yet.)

$\omega$ can be expressed from $\delta$ as

$$2\pi \cdot f_s \cdot \frac{j+\delta}{N} \quad (16)$$

The accuracy of estimators approaches the Cramer-Rao bound [9].

## 6 Conclusions

In this article we examined the possibilities, how to make sinusoidal model fitting to recorded data as stable and as fast as possible. It was shown that one step of the iterative model fitting can be made 25% faster in the case of the 2nd sinusoidal model, which uses amplitude and phase parameters instead of sine and cosine amplitudes. This model has the same convergence speed, if the starting parameter estimates are close enough to the optimal ones.

Since the four parameter sinusoidal fitting has at least one non-linear parameter, the popularly used Newton-Gauss iteration can be divergent. Numerical instabilities also can appear already at normal applications. The stability can be increased by using a "$\lambda$-method" that is the simplification of Levenberg-Marquardt method. Determination of $\lambda$ is also given. The L-M method seemed to be a few percent slower in the simulations than the simplified method.

A rarely used method is used for determination of starting values for amplitude, phase and frequency parameters that produces estimates close to the Cramer-Rao bound.

Starting value of DC offset is not critical and simply the average of the recorded samples can be used.

The created method is very fast due to the faster iteration steps and the very accurate initial parameters. The created method became very robust as well.

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