Integrated Circuits for Communication

Berkeley

Volterra/Wiener Representation of Non-Linear Systems

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Linear Input/Output Representation

- A linear (LTI) system is completely characterized by its impulse response function:

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

- causality $\Rightarrow \quad h(t) = 0; \; t < 0$
- y(t) has memory since it depends on

$$x(t-\tau); \quad \tau \in [-\infty, \infty]$$

Non-Linear Order-N Convolution

- Consider a degree-n system:

$$y_n(t) = \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} h_n(\tau_1, \tau_2, ..., \tau_n) x(t-\tau_1)...x(t-\tau_n) d\tau_1 ... d\tau_n$$

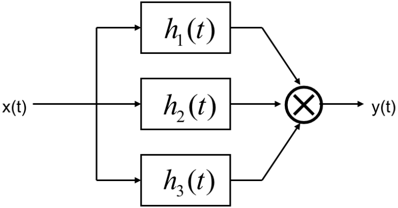
- If $x'(t) = \alpha x(t) \rightarrow y_n{}'(t) = \alpha^n y_n(t)$
- Change of variables -

$$\alpha_j = t - \tau_j \quad d\alpha_j = -d\tau_j$$
$$\tau_j = t - \alpha_j$$

Generalized Convolution

- Generalization of convolution integral of order n:
  $$y_n(t) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} h_n(\tau_1, \tau_2, \dots, \tau_n) x(t - \tau_1) \dots x(t - \tau_n) d\tau_1 \dots d\tau_n$$
- This describes a system that depends not only on the input $x(t)$ and all past values of the input $x(t - \tau)$, but also on powers of $x(t - \tau)$ where we take products between past values at different times.

Non-Linear Example



$$y_j(t) = \int_{-\infty}^{\infty} h_j(t-\tau)x(\tau)d\tau$$

$$y(t) = y_1(t)y_2(t)y_3(t)$$

$$= \int_{-\infty}^{\infty} h_1(\tau_1)x(t-\tau_1)d\tau_1 \cdot \int_{-\infty}^{\infty} \quad 2 \quad d\tau_2 \cdot \int_{-\infty}^{\infty} \quad 3 \quad d\tau_3$$

Non-Linear Example (cont)

$$= \int_{-\infty}^{\infty} h_1(\tau_1)x(t-\tau_1)d\tau_1 \cdot \int_{-\infty}^{\infty} \quad 2 \quad d\tau_2 \cdot \int_{-\infty}^{\infty} \quad 3 \quad d\tau_3$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(\tau_1)h_2(\tau_2)h_3(\tau_3)x(t-\tau_1)x(t-\tau_2)x(t-\tau_3)d\tau_1 d\tau_2 d\tau_3$$

$$h(t_1, t_2, t_3) = h_1(t_1)h_2(t_2)h_3(t_3)$$

$$h_s(\ ) = \frac{1}{6}\{h(t_1, t_2, t_3) + h(t_2, t_1, t_3) + h(t_2, t_3, t_1) + \ldots\}$$

- Kernel is not in unique. We can define a unique "symmetric" kernel as above.

Symmetry

- Kernel h can be expressed as a symmetric function of its arguments: Consider output of a system where we permute any number of indices of h:

$$\int_{-\infty}^{\infty} h(\tau_2,\tau_1)x(t-\tau_2)x(t-\tau_1)d\tau_2 d\tau_1$$
$$= \int_{-\infty}^{\infty} h(\tau_1,\tau_2)x(t-\tau_1)x(t-\tau_2)d\tau_1 d\tau_2$$

$$h(\tau_1,\tau_2) \Leftrightarrow h(\tau_2,\tau_1)$$

- For n arguments, $n!$ permutations

Symmetric kernel

- We create a symmetric kernel by

$$h_{sym}(t_1, ..., t_n) = \frac{1}{n!} \sum h\left(t_{\pi(1)}, ..., t_{\pi(n)}\right)$$

- System output identical to original unsymmetrical kernel
- Note that since the kernel is not unique, we are free to choose any valid kernel. The symmetric choice is one way to do it and it simplifies some of our calculations down the line.

Volterra Series

$$y(t) = \sum_{n=1}^{N} \int_{-\infty}^{\infty} h_n(\tau_1, ..., \tau_n) x(t - \tau_1) ... x(t - \tau_n) d\tau_1 ... d\tau_n$$

- Volterra Series: "Polynomial" of degree N
- If $h_n(t_1, ..., t_n) = a_n \delta(t_1)\delta(t_2)...\delta(t_n)$, we get an ordinary power series:

$$y(t) = a_1 x(t) + a_2 x(t)^2 + ... + a_N x(t)^N$$

- It can be rigorously shown by the Stone-Weierstrass theorem that the above polynomial approximates a non-linear system to any desired precision if N is made sufficiently large.

Non-rigorous "Proof"

- Say $y(t)$ is a non-linear function of $x(t-\tau)$ for all $\tau > 0$ (all past input)
- Fix time t and say that $x(t-\tau)$ can be characterized by the set $\{x_1(t), ..., x_n(t), ...\}$ so that $y(t)$ is some non-linear function:

$$y(t) = f(x_1(t), x_2(t), ...)$$

Non-Rigorous Proof (cont)

- Let $\{\varphi_1(t), \varphi_2(t), \ldots\}$ be an orthonormal basis for the space

$$\int_{-\infty}^{\infty} \varphi_i(\tau)\varphi_j(\tau)d\tau = \delta_{ij}$$

- Thus

$$x(t-\tau) = \sum_{i=1}^{\infty} x_i(t)\varphi_i(\tau)$$

- "inner product"

$$x_i(t) = \int_{-\infty}^{\infty} x(t-\tau)\varphi_i(\tau)d\tau$$

Non-Rigorous Proof (cont)

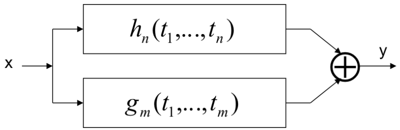
- Expand f into a Taylor series :

$$f\left(x_1(t), x_2(t), \ldots\right)$$

$$y(t) = a_o + \sum_{i=1}^{\infty} a_i x_i(t) + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_i a_j x_i(t) x_j(t) + \ldots$$
$$= a_o + \int_0^{\infty} \sum_{i=1}^{\infty} a_i \varphi(\tau_1) x(t-\tau_1) d\tau_1 + \int_0^{\infty}\int_0^{\infty} \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} \varphi_i(\tau_1) \varphi_j(\tau_2) x(t -$$

- This is the Volterra/Wiener representation for a non-linear system
- Sifting Property: $x(\sigma) = \int_{-\infty}^{\infty} \delta(t-\sigma) x(t) dt$

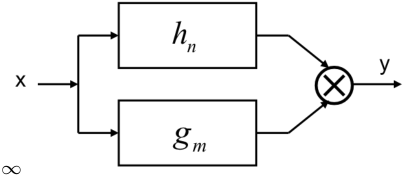
Interconnection of Non-Linear Systems



- Sum:

$$f_n(t_1, t_2, \ldots, t_n) = h_n(t_1, \ldots, t_n) + g_m(t_1, \ldots, t_m)$$

Product Interconnection



$$y(t) = \int_{-\infty}^{\infty} h_n(\tau_1, ..., \tau_n)x(t-\tau_1)...x(t-\tau_n)d\tau_1...d\tau_n \times$$

$$\int_{-\infty}^{\infty} g_m(\tau_1, ..., \tau_m)x(t-\tau_1)...x(t-\tau_m)d\tau_1...d\tau_m$$

$$= \int_{-\infty}^{\infty} h_n(\tau_1, ..., \tau_n)g_m(\tau_{n+1}, ..., \tau_{n+m})x(t-\tau_1)...x(t-\tau_{n+m})d\tau_1...d\tau_{n+m}$$

$$f_{n+m}(t_1, ..., t_{n+m}) = h_n(t_1, ..., t_n)g_m(t_{n+1}, ..., t_{n+m})$$

Volterra Series Laplace Domain

- Transform domain input/output representation
- Linear systems in time domain

$$F(s) = L[f(t)] = \int_0^\infty f(t)e^{-st}dt$$

- Define Generalized Laplace Transform:

$$F(s_1, ..., s_n) = L[f(t_1, ..., t_n)]$$
$$= \int_0^\infty f(t_1, ..., t_n)e^{-s_1 t_1} \cdots e^{-s_n t_n} dt_1 \cdots dt_n$$

Volterra Series Example

- Generalized transform of a function of two variables:

$$f(t_1, t_2) = t_1 - t_1 e^{-t_2} \quad t_1, t_2 \geq 0$$

$$F(s_1, s_2) = \int_0^\infty \int_0^\infty t_1 e^{-s_1 t_1} e^{-s_2 t_2} dt_1 dt_2 - \int_0^\infty \int_0^\infty t_1 e^{-t_2} e^{-s_1 t_1} e^{-s_2 t_2} dt_1 dt_2$$

$$F(s_1, s_2) = \frac{1}{{s_1}^2} \left(\int_0^\infty e^{-s_2 t_2} dt_2 - \int_0^\infty e^{-t_2} e^{-s_2 t_2} dt_2 \right)$$

$$= \frac{1}{{s_1}^2 s_2 (s_2 + 1)}$$

Properties of Transform

- Property 1: L is linear
- Property 2:

$$f(t_1, ..., t_n) = h(t_1, ..., t_k)g(t_{k+1}, ..., t_n)$$
$$\Leftrightarrow$$
$$F(s_1, ..., s_n) = H(s_1, ..., s_k)G(s_{k+1}, ..., s_n)$$

- Property 3: Convolution form #1

$$f(t_1, ..., t_n) = \int_0^\infty h(\tau)g(t_1 - \tau, ..., t_n - \tau)d\tau$$
$$F(s_1, ..., s_n) = H(s_1 + \cdots + s_n)G(s_1, ..., s_n)$$

Properties of Generalized Transform

- Property 4: Convolution Form #2:

$$f(t_1,...,t_n) = \int_0^\infty h(t_1-\tau_1,...,t_n-\tau_n)g(\tau_1,...,\tau_n)\times d\tau_1...d\tau_n$$

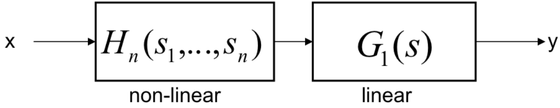
$$F(s_1,...,s_n) = H(s_1,...,s_n)G(s_1,...,s_n)$$

- Property 5: Time delay $\tau_j > 0$

$$L[f(t_1-\tau_1,...,t_n-\tau_n)] = F(s_1,...,s_n)e^{-s_1\tau_1...s_n\tau_n}$$

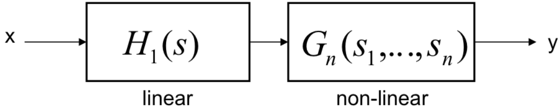
Cascades of Systems

- Cascade #1:



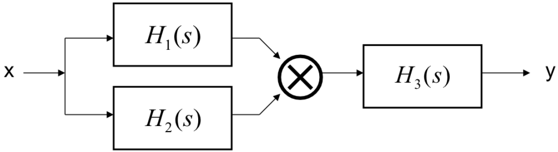
$$F_n(s_1, \ldots, s_n) = H_n(s_1, \cdots s_n) G_1(s_1 + \cdots + s_n)$$

- Cascade #2:



$$F_n(s_1, \ldots, s_n) = H_1(s_1) \cdots H_1(s_n) G_n(s_1, \ldots, s_n)$$

Cascade Example



$$F(s_1, s_2) = H_1(s_1)H_2(s_2)H_3(s_1 + s_2)$$

- By property #1 and property #2
- Note that H is not symmetric

Impulse Response of Non-Linear Volterra System

- Suppose we apply two impulses into an n'th order system

$$y(t) = \int_{-\infty}^{\infty} h_n(\tau_1, \tau_2, \cdots) x(t-\tau_1) x(t-\tau_2) \cdots d\tau_1 d\tau_2 \cdots$$

- Using the sifting property of the delta functions, we have

$$y(t) = h_n(t, t, \cdots)$$

“Two Impulse Response” of Non-Linear Volterra System

- Suppose we apply two impulses into a second order system

$$x(t) = \delta(t) + \delta(t - T)$$

- Taking the product of $x(t_1) \cdot x(t_2)$ gives four products

$$\delta(t_1)\delta(t_2) + \delta(t_1 - T)\delta(t_2 - T) + \delta(t_1 - T)\delta(t_2) + \delta(t_1)\delta(t_2 - T)$$

- After integration, we have an interesting result

$$\int_{-\infty}^{\infty} h_2(\tau_1, \tau_2) x(t - \tau_2) x(t - \tau_1) d\tau_1 d\tau_2$$

$$= h_2(t, t) + h_2(t - T, t) + h_2(t, t - T) + h_2(t - T, t - T)$$

- With two delta inputs, and by varying their relative delay, we can find the second order non-linearity of a system

Exponential Response of n-th Order System

$$y_n(t) = \int_{-\infty}^{\infty} h_n(\tau_1, ..., \tau_n) x(t - \tau_1) \cdots x(t - \tau_n) d\tau_1 ... d\tau_n$$

$$x(t) = \sum_{i=1}^{N} \alpha_i e^{\lambda_i t}$$

$$y_n(t) = \int_{-\infty}^{\infty} h_n(\tau_1, ..., \tau_n) \prod_{j=1}^{n} [\alpha_1 e^{\lambda_1 (t - \tau_j)} + \cdots + \alpha_p e^{\lambda_p (t - \tau_j)}] \times d\tau_1 \cdots d\tau_n$$

$$\left(\sum_{k_1=1}^{N} \alpha_{k_1} e^{\lambda_{k_1} (t - \tau_1)} \right) ... \left(\sum_{k_n=1}^{N} \alpha_{k_n} e^{\lambda_{k_n} (t - \tau_n)} \right)$$

Exponential Response (cont)

$$\sum_{k_1=1}^{N} \cdots \sum_{k_n=1}^{N} \left(\prod_{j=1}^{n} \alpha_{k_j} \right) \exp\{\lambda_{k_1}(t-\tau_1) + \ldots + \lambda_{k_n}(t-\tau_n)\}$$

$$\exp\{\sum_{j=1}^{n} \lambda_{k_j}(t-\tau_j)\}$$

$$y_n(t) = \sum_{k_1=1}^{N} \cdots \sum_{k_n=1}^{N} \left(\prod_{j=1}^{n} \alpha_{k_j} \right) \exp\{\sum_{j=1}^{n} \lambda_{k_j} t\} \int_{-\infty}^{\infty} h_n(\tau_1, \ldots \tau_n) \exp\{-\sum_{j=1}^{n} \lambda_{k_j} \tau_j\} d\tau_1 \ldots d\tau_n$$

$$H_n(\lambda_{k_1}, \ldots, \lambda_{k_n})$$

Collecting Terms

$$y_n(t) = \sum_{k_1=1}^{N} \dots \sum_{k_n=1}^{N} \left(\prod_{j=1}^{n} \alpha_{k_j} \right) \exp\{\sum_{l=1}^{n} \lambda_{k_l} t\} H_n \left(\lambda_{k_1}, \dots, \lambda_{k_n}\right)$$

- We've seen this before ... A particular frequency mix $m_1\lambda_1 + m_2\lambda_2 + \dots + m_p\lambda_p$ has response

$$\alpha_1{}^{m_1} \dots \alpha_p{}^{m_p} G_{m_1,\dots,m_p}(\lambda_1, \dots, \lambda_n) e^{(m_1\lambda_1 + \dots + m_p\lambda_p)t}$$

- where the function G takes into account how many times a particular mix occurs...[1]

[1] See the lecture module on multi-tone input into a memoryless non-linear power series for a review.

Frequency Mix Vector $\vec{k}$

$$y_n(t) = \sum_{\vec{k}} {\alpha_1}^{m_1} ... {\alpha_p}^{m_p} G_{\vec{k}}(\vec{\lambda}) \exp\{\vec{m} \cdot \vec{x}\}$$

- Sum over all vectors $\vec{k}$ such that $0 \leq k_i \leq n$ and $\sum_j k_j = N$
- If $H_n(s_1, ..., s_n)$ is symmetric, then we can group the terms as before

$$G_{\vec{k}}(\vec{\lambda}) = (n; \vec{k}) H_{n,sym}(\underbrace{\lambda_1, ..., \lambda_1}_{k_1}, ..., \underbrace{\lambda_p, ..., \lambda_p}_{k_p})$$

Important special case $N = n$

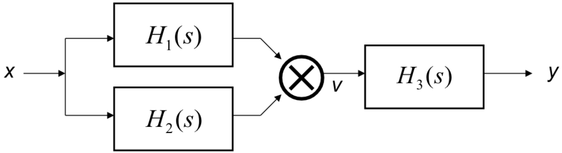
$$G_{\vec{k}}(\lambda_1, ..., \lambda_n) = n! H_{n,sym}(\lambda_1, ..., \lambda_n)$$

- To derive $H_{n,sym}(\lambda_1, ..., \lambda_n)$, we can apply n exponentials to a degree n system and the symmetric transfer function is given by $\frac{1}{n!}$ times the coefficient of

$$e^{\lambda_1 t + ... + \lambda_n t}$$

- We call this the "Growing Exponential Method"

Example 3



- Excite system with two-tones:

$$v(t) = \left(H_1(\lambda_1)e^{\lambda_1 t} + H_1(\lambda_2)e^{\lambda_2 t}\right) \times \left(H_2(\lambda_1)e^{\lambda_1 t} + H_2(\lambda_2)e^{\lambda_2 t}\right)$$

$$= H_1(\lambda_1)H_2(\lambda_1)e^{2\lambda_1 t} + H_1(\lambda_2)H_2(\lambda_2)e^{2\lambda_2 t}$$
$$+ H_1(\lambda_1)H_2(\lambda_2)e^{(\lambda_1+\lambda_2)t} + H_1(\lambda_2)H_2(\lambda_1)e^{(\lambda_1+\lambda_2)t}$$

Example 3 (cont)

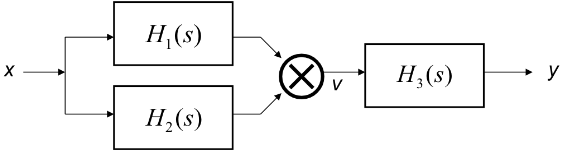
$$y(t) = H_1(\lambda_1)H_2(\lambda_1)H_3(2\lambda_1)e^{2\lambda_1 t} + H_1(\lambda_2)H_2(\lambda_2)H_3(2\lambda_2)e^{2\lambda_2 t}$$
$$+[H_1(\lambda_1)H_2(\lambda_2) + H_1(\lambda_2)H_2(\lambda_1)] \cdot H_3(\lambda_1 + \lambda_2)e^{(\lambda_1+\lambda_2)t}$$

$$2!\ H_{sym}(s_1, s_2)$$

$$H_{sym}(s_1, s_2) = \frac{1}{2}[H_1(s_1)H_2(s_2) + H_1(s_2)H_2(s_1)]H_3(s_1 + s_2)$$

Example 4

- Non-linear system in parallel with linear system:



$$y_1 = \int_{-\infty}^{\infty} h_1(\tau_1)x(t-\tau_1)d\tau_1$$

$$y_2 = \int_{-\infty}^{\infty} h_2(\tau_2,\tau_3)x(t-\tau_2)x(t-\tau_3)d\tau_2 d\tau_3$$

$$y_1 \times y_2 = \int_{-\infty}^{\infty} h_1(\tau_1)h_2(\tau_2,\tau_3)x(t-\tau_1)\cdots x(t-\tau_3)d\tau_1 ... d\tau_3$$

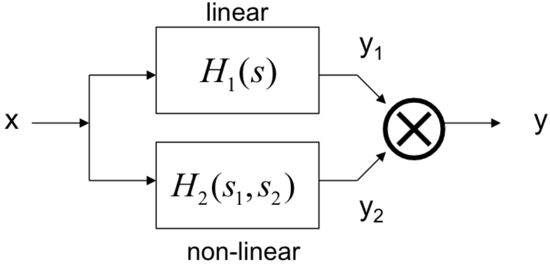
Example 4 (cont)

$$H_{sym}(s_1, s_2, s_3) = \frac{1}{3}\{H_1(s_1)H_2(s_2, s_3) + H_1(s_2)H_2(s_1, s_3)$$
$$+H_1(s_3)H_2(s_1, s_2)\}$$

- assuming H_2 is symmetric
- Notation:

$$H_{sym}(s_1, s_2, s_3) = \overline{H_1(s_1)H_2(s_2, s_3)}$$

Example 4 Again



- Redo example with growing exponential method
- Overall system is third order, so apply sum of 3 exponentials to system

$$e^{\lambda_1 t} + e^{\lambda_2 t} + e^{\lambda_3 t}$$

Example 4 Again (cont)

- We can drop terms that we don't care about
- We only care about the final term $e^{\lambda_1 t} + e^{\lambda_2 t} + e^{\lambda_3 t}$, so for now ignore terms except $e^{(\lambda_j + \lambda_k)t}$ where $j \neq k$
- Focus on terms in y_2 first:

$$
\begin{gathered}
2e^{(\lambda_1+\lambda_2)t} H_{2s}(\lambda_1, \lambda_2) \\
2e^{(\lambda_1+\lambda_2)t} H_{2s}(\lambda_1, \lambda_3) \\
2e^{(\lambda_2+\lambda_3)t} H_{2s}(\lambda_2, \lambda_3)
\end{gathered}
$$

$$
H_{2s}(\lambda_1, \lambda_2) = H_{2s}(\lambda_2, \lambda_1)
$$

Example 4 again (cont)

- Now the product of $y_1(t)$ and $y_2(t)$ produces terms like $e^{(\lambda_1+\lambda_2+\lambda_3)t}$

$$
\begin{aligned}
&2H_{2s}(\lambda_1,\lambda_2)H_1(\lambda_3)e^{(\lambda_1+\lambda_2+\lambda_3)t}\\
&+2H_{2s}(\lambda_1,\lambda_3)H_1(\lambda_2)e^{(\lambda_1+\lambda_2+\lambda_3)t}\\
&+2H_{2s}(\lambda_2,\lambda_3)H_1(\lambda_1)e^{(\lambda_1+\lambda_2+\lambda_3)t}\\
&=\ 3!\,H_{3s}(\lambda_1,\lambda_2,\lambda_3)
\end{aligned}
$$

$$
H_{3s}(s_1,s_2,s_3)=\frac{2}{3!}\left(\qquad\qquad\right)
$$

Singe-Tone Sinusoidal Response

- Let's warm up by calculating the n'th order response to a sinusoidal input

$$y(t) = \int_{-\infty}^{\infty} h_{sym(\sigma_1,\cdots,\sigma_n)} \prod_{j=1}^{n} \frac{e^{j\omega(t-\sigma_j)} + e^{-j\omega(t-\sigma_j)}}{2} d\sigma_1 \cdots d\sigma_n$$

- Let's group the exponentials by using the following notation: $\lambda_1 = j\omega$ and $\lambda_2 = -j\omega$. Expanding the product as sums

$$\prod = \frac{1}{2^n} \sum_{k_1=1}^{2} \cdots \sum_{k_n=1}^{2} \left(\sum_{j=1}^{n} \exp(\lambda_{k_j}(t - \sigma_j)) \right)$$

Sinusoidal Response (invoke Laplace)

- We can now simplify the expression by noting that the

$$\int_{-\infty}^{\infty} h_{sym}(\sigma_1, \cdots, \sigma_n) \exp(-\sum_{j=1}^{n} \lambda_{k_j} \sigma_j) d\sigma_1 \cdots d\sigma_n = H(\lambda_{k_1}, \cdots, \lambda_{k_n})$$

$$= \frac{1}{2^n} \sum_{k_1=1}^{2} \cdots \sum_{k_n=1}^{2} \exp(\sum_{j=1}^{n} \lambda_{k_j} t) H(\lambda_{k_1}, \cdots, \lambda_{k_n})$$

- A particular term has a frequency given by $k\lambda_1 + (n-k)\lambda_2 = (2k-n)\omega$

$$= \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} e^{j(2k-n)\omega t} H(\underbrace{\omega, \cdots, \omega}_{k}, \underbrace{-\omega, \cdots, -\omega}_{n-k})$$

$$= \frac{1}{2^n} \sum_{k=0}^{n} G_{k,n-k}(j\omega, -j\omega) e^{j(2k-n)\omega t}$$

- Just as expected, an nth order system generates the n'th harmonic and every other harmonic down to either DC (n is even) or fundamental (n is odd).

Grouping Terms

- We can group positive and negative frequency terms together by noting that

$$G_{k,n-k}(j\omega,-j\omega) = \begin{pmatrix} n \\ k \end{pmatrix} H_{sym}(\underbrace{\omega,\cdots,\omega}_{k},\underbrace{-\omega,\cdots,-\omega}_{n-k})$$

$$G_{n-k,k}(-j\omega,j\omega) = \begin{pmatrix} n \\ n-k \end{pmatrix} H_{sym}(\underbrace{-\omega,\cdots,-\omega}_{n-k},\underbrace{\omega,\cdots,\omega}_{k})$$

$$\begin{pmatrix} n \\ k \end{pmatrix} = \begin{pmatrix} n \\ n-k \end{pmatrix}$$

- By the symmetry of the kernel and the binomial coefficient, we have

$$G_{n-k,k}(-j\omega,j\omega) = G_{k,n-k}(j\omega,-j\omega)$$

Sinusoidal Response Summary

- We can now write the total sinusoidal response as

$$
\begin{aligned}
& y_n(t) = \frac{1}{2^{n-1}} \left| G_{n,0}(j\omega, -j\omega) \right| \cos(n\omega t + \angle G_{n,0}(j\omega, -j\omega)) + \\
& \frac{1}{2^{n-1}} \left| G_{n-1,1}(j\omega, -j\omega) \right| \cos((n-2)\omega t + \angle G_{n-1,1}(j\omega, -j\omega)) + \cdots + \\
& \begin{cases} \frac{1}{2^{n-1}} \left| G_{n/2,n/2}(j\omega, -j\omega) \right| & \text{n even} \\ \frac{1}{2^{n-1}} \left| G_{\frac{n+1}{2},\frac{n+1}{2}}(j\omega, -j\omega) \right| \cos(\omega t + \angle G_{\frac{n+1}{2},\frac{n+1}{2}}(j\omega, -j\omega)) & \text{n odd} \end{cases}
\end{aligned}
$$

- We need to form the above some for each power in the Volterra series.

Sinusoidal Multi-Tone Response, n'th power only

- Calculation is very similar to exponential response, but now we just need to keep track of complex conjugate frequency. Using our shorthand notation $\omega_{-k} = -\omega_k$ and $A_0 \equiv 0$

$$x(t) = \sum_{k=0}^{N} A_k \cos(\omega_k t) = \frac{1}{2} \sum_{k=-N}^{N} A_k e^{j\omega_k t}$$

$$y(t) = \int_{-\infty}^{\infty} h_{sym}(\sigma_q, \cdots, \sigma_n) \prod_{i=1}^{n} \left(\sum_{k=-N}^{N} \frac{A_k}{2} e^{j\omega_k (t-\sigma_i)} \right) d\sigma_1 \cdots d\sigma_n$$

$$\frac{1}{2^n} \left(\sum_{k_1=-N}^{N} A_{k_1} e^{j\omega_{k_1}(t-\sigma_1)} \right) \times \left(\sum_{k_2=-N}^{N} A_{k_2} e^{j\omega_{k2}(t-\sigma_2)} \right) \times \cdots$$

Product of Sums as Sum of Products

- Expanding the product of sums, we can sum over all possible vectors $\vec{k}$

$$= \sum_{\vec{k}} G_{\vec{k}} \; e^{j\vec{k}\cdot\vec{\omega}t} \; e^{j\vec{k}\cdot\vec{\sigma}t}$$

- The term $G_{\vec{k}}$ is used to collect all frequency products that sum to the same frequency, determined by the vector $\vec{k}$:

$$G_{\vec{k}} = \frac{1}{2^n}(n;\vec{k})A_1^{k_1} \cdot A_2^{k_2} \cdots A_n^{k_n}$$

- Where we have already met the multi-nomial coefficient $(n;\vec{k})$ to account for the number of times a particular frequency product occurs.

Sinusoidal Response (cont)

- Performing the integration, we observe that

$$\int_{-\infty}^{\infty} h_{sym}(\sigma_1, \cdots, \sigma_n) e^{-j(\omega_{k_1}\sigma_1 + \cdots + \omega_{k_n}\sigma_n)} d\sigma_1 \cdots d\sigma_n$$

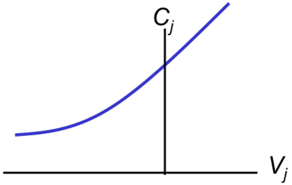
$$= H(\omega_{k_1}, \cdots, \omega_{k_n})$$

- Keep in mind the symmetry of the multi-nomial coefficient and the fact that every positive frequency term is accompanied by a negative counterpart obtained by inverting the $\vec{k}$ vector.
- This allows us to write the final sinusoidal sum as.

$$\sum_{\vec{k}} (n; \vec{k}) \frac{A_1^{k_1} \cdots A_n^{k_n}}{2^{n-1}} \left| H(\omega_{k_1}, \cdots, \omega_{k_n}) \right| \cos(\vec{k} \cdot \vec{\omega} t + \angle H(\omega_{k_1}, \cdots, \omega_{k_n}))$$

Capacitive non-linearity

- Non-linear capacitors:
  - BJT: C_μ and C_{cs}
  - FET: C_{db} and C_{sb}



- Small signal (incremental) capacitance ($n \approx 2-3$)

$$C_j = \frac{dQ}{dV_j} = \frac{K}{(\Phi + V_j)^{\frac{1}{n}}}$$

$$\text{Let } V_j = V_Q + v$$

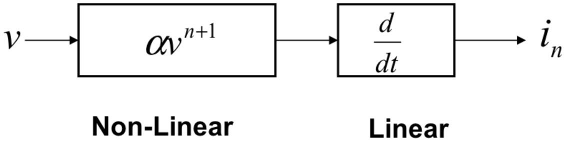
$$C_j = \frac{K}{(\Phi + V_j)^{\frac{1}{n}} (1 + \frac{v}{\Phi + V_Q})^{\frac{1}{n}}} \approx C_{\mu_o} + C_{\mu_1} v + C_{\mu_2} v^2 + \ldots$$

Cap Non-Linearity (cont)

$$i = \frac{dQ}{dt} = \frac{dQ}{dv}\frac{dv}{dt} = C_j(v)\frac{dv}{dt}$$

$$\begin{aligned} i &= C_{\mu 0}\frac{dv}{dt} + C_{\mu_1} v \frac{dv}{dt} + C_{\mu_2} v^2 \frac{dv}{dt} + \cdots \\ &= C_{\mu 0}\frac{dv}{dt} + \frac{C_{\mu 1}}{2}\frac{dv}{dt}^2 + \frac{C_{\mu 2}}{3}\frac{dv}{dt}^3 + \cdots \end{aligned}$$

- Model:

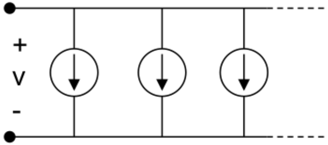


Overall Model

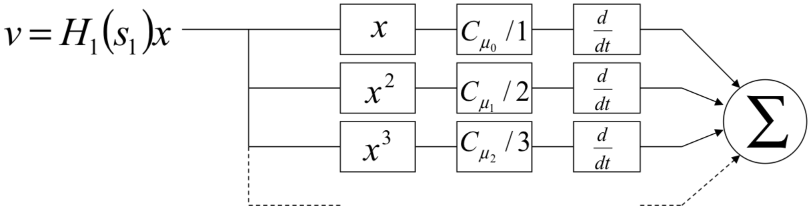
$$i = \frac{dQ}{dt} = \frac{dQ}{dv}\frac{dv}{dt} = C_j(v)\frac{dv}{dt}$$

$$C_j(v) = C_{\mu_o} + C_{\mu_1} v + C_{\mu_2} v^2 + \ldots$$

$$i = C_{\mu_o}\frac{dv}{dt} + \frac{1}{2} C_{\mu_1}\frac{dv^2}{dt} + \frac{1}{3} C_{\mu_2}\frac{dv^3}{dt} + \ldots$$



Cap Model Decomposition



- Let

$$v = H_1(s_1)\,x$$

$$v^2 = H_1(s_1)\,H_1(s_2)\,x^2$$

$$i_2 = (s_1 + s_2)\,H_1(s_1)\,H_1(s_2)\,\frac{1}{2}C_{\mu_1}x^2$$

$$i_n = (s_1 + \ldots + s_n)\,H_1(s_1)\ldots H_1(s_n)\,\frac{1}{n}C_{\mu,n-1}x^n$$

Volterra Operator Notation

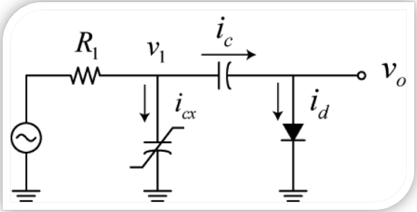
- The operation $\circ$ will be used as a shorthand notation

$$H(j\omega_1, j\omega_2, \cdots) \circ v^k$$

- The above equation implies that the coefficient H must be evaluated at the distortion product(s) of the signal v^k.
- Thus, the generalized power series is written in this form

$$v_0 = H_1(j\omega) \circ v_i + H_2(j\omega_1, j\omega_2) \circ v_i^2 + \cdots + H_k(j\omega_1, j\omega_2, \cdots) \circ v^k$$

A Real Circuit Example



(Note: DC Bias not shown)

- Find distortion in v_o for sinusoidal steady state response

$$v_o = B_1(j\omega_1) \circ v_i + B_2(j\omega_1, j\omega_2) \circ {v_i}^2 + \ldots$$

- Need to also find

$$v_1 = A_1(j\omega_1) \circ v_i + A_2(j\omega_1, j\omega_2) \circ {v_i}^2 + \ldots$$

Circuit Example (cont)

- Setup non-linearities
- Diode:

$$i_d = I_S e^{(v_o + V_Q)/V_T} - I_Q$$
$$= I_S e^{V_Q/V_T} e^{v_o/V_T} - I_Q = I_Q \left(e^{v_o/V_T} - 1 \right)$$

$$= g_1 v_o + g_2 {v_o}^2 + \ldots$$

- Capacitor:

$$C_x = \frac{dQ}{dv_1} = C_o + C_1 v_1 + C_2 {v_1}^2 + \ldots$$

$$i_{cx} = C_o \frac{dv_1}{dt} + \frac{C_1}{2} \frac{d{v_1}^2}{dt} + \frac{C_2}{3} \frac{d{v_1}^3}{dt}$$

Second-Order Terms

$$0 = \frac{A_2}{R_1} + j\left(\omega_a + \omega_b\right) C\left(A_2 - B_2\right) +$$

$$j\left(\omega_a + \omega_b\right) C_o A_2 + j\left(\omega_a + \omega_b\right) \frac{C_1}{2} A_1\left(j\omega_o\right) A_1\left(j\omega_b\right)$$

$$-j\left(\omega_a + \omega_b\right) C\left(A_2 - B_2\right) + g_1 B_2\left(j\omega_a, j\omega_b\right) +$$

$$g_2 B_1\left(j\omega_a\right) B_1\left(j\omega_b\right) = 0$$

- Solve for A and B

Third-Order Terms

$$\frac{A_3}{R_1} + j(\omega_a + \omega_b + \omega_c) C (A_3 - B_3) + j(\omega_a + \omega_b + \omega_c) C_o A_3 +$$

$$j(\omega_a + \omega_b + \omega_c) \frac{C_1}{2} 2\overline{A_1(j\omega_a) A_2(j\omega_a, j\omega_b)} +$$

$$j(\omega_a + \omega_b + \omega_c) \frac{C_2}{3} A_1(j\omega_a) A_1(j\omega_b) A_1(j\omega_c) = 0$$

$$-j(\omega_a + \omega_b + \omega_c) C (A_3 - B_3) + g_1 B_3 + g_2 2\overline{B_1 B_2} +$$

$$g_3 B_1 B_1 B_1 = 0$$

- Solve for A_3 and B_3

Distortion Calc at High Freq

$$s_o = H_1(j\omega_a) \circ s_i + H_2(j\omega_a, j\omega_b) \circ {s_i}^2 + \ldots$$

- Compute IM_3 at $\omega_2 - \omega_1$ only generated by $n = 3$

$$\vec{k}_{IM_3} = (0 \quad 0 \quad 1 \quad 0 \quad 2 \quad 0)$$

- H_3 is symmetric so we can group all terms producing this frequency mix by H_3

$$\frac{\left(3; \vec{k}_{IM_3}\right)}{2^{3-1}} = \frac{3!}{2! \cdot 4} = \frac{3}{4} \qquad \frac{3}{4} H_3(j\omega_2, j\omega_2, -j\omega_1) s_1 s_2^2$$

$$IM_3 = \frac{3}{4} \frac{s_1 {s_2}^2 \left|H_3(j\omega_2, j\omega_2, -j\omega_1)\right|}{\left|H_1(j\omega_1)\right| s_1}$$

- For equal amp o/p signal, we adjust each input amp so that:

$$s_o = \left|H_1(j\omega_1)\right| s_1 = \left|H_1(j\omega_2)\right| s_2$$

Disto Calc at High Freq (2)

$$IM_3 = \frac{3}{4} \frac{|H_3(j\omega_2, j\omega_2, -j\omega_1)|}{|H_1(j\omega_1)| \, |H_1(j\omega_2)|^2} {s_o}^2$$

- At low frequency:

$$IM_3 = \frac{3}{4} \frac{a_3}{{a_1}^3} {s_o}^2$$

- Conclude that at high frequency all third order distortion (fractional) $\propto$ (signal level)^2 for small distortion.
- All second order $\propto$ (signal level)

Disto Calc at High Freq (3)

- Similarly

$$HD_3 = \frac{{s_1}^3}{4} \frac{|H_3(j\omega_1, j\omega_1, j\omega_1)|}{s_o}$$

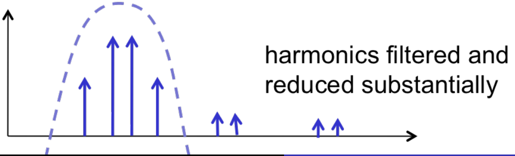
$$s_o = |H_1(j\omega_1)|\, s_1$$

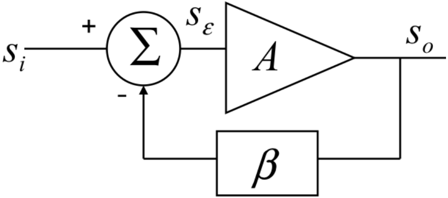
$$HD_3 = \frac{1}{4} \frac{|H_3(j\omega_1, j\omega_1, j\omega_1)|}{|H_1(j\omega_1)|^3} {s_o}^2$$

- Low Freq.

$$HD_3 = \frac{1}{4} \frac{a_3}{{a_1}^3} {s_o}^2$$

- No fixed relation between HD_3 and IM_3





Let

$$s_o = A_1(j\omega_a) \circ s_2 + A_2(j\omega_a, j\omega_b) \circ {s_\varepsilon}^2 + \ldots$$

$$s_{fb} = \beta(j\omega_a) \circ s_o \qquad\qquad s_\varepsilon = s_i - s_{fb}$$

- Look for

$$s_o = B_1(j\omega_a) \circ s_i + B_2(j\omega_a, j\omega_b) \circ {s_i}^2 + \ldots$$

$$B_1(j\omega_a) \circ s_i + B_2 \circ {s_i}^2 + \ldots = A_1\left(s_i - \beta(j\omega_a) \circ \left(B_1(j\omega_a) \circ s_i + B_2 \circ {s_i}^2 + \ldots\right) + \ldots\right) + A_2 \circ ()^2 + \ldots$$

High Freq Disto & FB (2)

- First order: $B_1(j\omega_a) = \frac{A_1(j\omega_a)}{1+A_1(j\omega_a)\beta(j\omega_a)}$
- Second order:

$$B_2 = -A_1(j\omega_a + j\omega_b)\,\beta(j\omega_a + j\omega_b)\,B_2(j\omega_a, j\omega_b) + A_2(j\omega_a, j\omega_b)\,B_1(j\omega_a)\,B_1(j\omega_b)$$

$$B_2(j\omega_a, j\omega_b) = \frac{A_2(j\omega_a, j\omega_b)\,B_1(j\omega_a)\,B_1(j\omega_b)}{(1 + A_1(j\omega_a + j\omega_b)\,\beta(j\omega_a + j\omega_b)\,B_2(j\omega_a, j\omega_b))}$$

$$B_2(j\omega_a, j\omega_b) = \frac{A_2(j\omega_a, j\omega_b)}{[1 + A_1(j\omega_a + j\omega_b)\,\beta(j\omega_a + j\omega_b)]\times}$$

$$[1 + A_1(j\omega_a)\,\beta(j\omega_a)] \times [1 + A_1(j\omega_b)\,\beta(j\omega_b)]$$

Comments about HF/LF Disto

- Feedback reduces distortion at low frequency and high frequency $\times \frac{1}{1+T}$ for a fixed output signal level
- True at high frequency if we use $\left|\frac{1}{1+T(j\omega)}\right|$ where ω is evaluated at the frequency of the distortion product
- While IM/HD no longer related, CM, TB, P_{-1dB}, P_{BL} are related since frequencies close together
- Most circuits (90%) can be analyzed with a power series

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