Active Learning in Regression Over Finite Domains

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Based on: [Antos et al., 2010, Carpentier et al., 2011, Carpentier et al., 2014], PDSS, IDA jegyzet

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   - Bayes-döntés
   - Bayes-döntés közelítése
   - Bayes-becslés
   - Regresszióbecslés; négyzetes középhiba minimalizálás
2. Introduction
3. Model, Protocol, Losses
4. Max-Loss
   - Algorithm — GAFS-MAX
   - Theory
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   - Theory
   - Experiments
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5. Conclusions

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KÖVETKEZTETÉS

- Megfigyelhető X-ből kell egy (még) nem megfigyelhető Y-ra következtetni
- X, Y val. változók, értékkészletük $\mathcal{X}$ ill. $\mathcal{Y}$ (pl. $\mathbb{R}$, $\mathbb{R}^d$, $\{0,1\}^d$, $[0,1]^d$, ezek megszámlálható v. véges része)
- $\mathcal{Y}$ elemei: *címkék*
- X-ből Y-ra a $g : \mathcal{X} \to \bar{\mathcal{Y}}$-vel következtetünk ($\bar{\mathcal{Y}} (\supseteq \mathcal{Y})$ az szóba jövő g-k értékkészletének uniója)
- A $g(X)$ *következtetés* jóságát méri: $C : \mathcal{Y} \times \bar{\mathcal{Y}} \to [0, \infty)$ *költségfüggvény*
- $C(y, g(x))$ a költsége, hogy a valódi $Y = y$ helyett g az $X = x$-re $g(x)$-re következtet

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GLOBÁLIS, LOKÁLIS KOCKÁZAT

- g jóságát $R(g) \stackrel{\text{def}}{=} \mathbb{E}\left[C(Y, g(X))\right]$ *globális kockázat* méri
- g-k, amikre $R(g)$ a legkisebb: *optimális*
- $r(g,x) \stackrel{\text{def}}{=} \mathbb{E}\left[C(Y, g(X)) | X = x\right]$: *lokális kockázat függvény*
- $\mathbb{E}\left[r(g,X)\right] = R(g)$ és

$$r(g,x) = \int_y C(y, g(x))\, dF_{Y|x}(y)$$

ahol $F_{Y|x}$ az Y feltételes, *a posteriori eloszlásfüggvénye* ha $X = x$.

(Általános integrál: diszkrét e.o. $\Rightarrow$ összegzés, absz. folyt. e.o.$\Rightarrow$ Riemann-integrál)

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Döntés, becslés, gyakori költségfüggvények

- Ha $\forall y \in \mathcal{Y}$-ra $C(y,\cdot)$ az $\bar{\mathcal{Y}} \setminus \{y\}$-on konstans $\Rightarrow$ *döntési probléma*
  - Általában $\bar{\mathcal{Y}} = \mathcal{Y}$ diszkrét (megszámlálható), pl. $\mathcal{Y} = \{0,1\}$
  - tipikus C: $C_0(y,y') \stackrel{\text{def}}{=} \mathbb{I}_{\{y \neq y'\}}$, *0-1* költség ($\mathbb{I}_A$ az A indikátorfüggvénye)
  - $\Rightarrow R(g) = \mathbb{P}(Y \neq g(X)) = g$ hibázásának valószínűsége
- Ha C valami intuitív távolság $\Rightarrow$ *becslési probléma*

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HIPOTÉZIS, DÖNTÉSFÜGGVÉNY, DÖNTÉSI TARTOMÁNY

Spec. eset: $\mathcal{Y} = \bar{\mathcal{Y}} = \{0, 1\}$, $C(i, j) = C_0(i, j) = \mathbb{I}_{\{i \neq j\}}$

DEFINITIONS

$i = 0{,}1$ az $\{Y = i\}$: *i-dik hipotézis*. Y a posteriori eloszlását az $\eta_i(x) \stackrel{\text{def}}{=} \mathbb{P}(Y = i | X = x)$ *a posteriori valószínűségek* adják. $g =$*döntésfüggvény*. 0 és 1 g-vel való ősképei $\mathcal{X}$ egy partíciója, ennek $D_i = \{x \in \mathcal{X} : g(x) = i\}$ osztályai a *döntési tartományok*.

$(D_0, D_1) \Leftrightarrow g$, mert $\mathbb{I}_{\{g(x)=j\}} = \mathbb{I}_{\{x \in D_j\}}$. Így $r(g, x) =$

$$\mathbb{I}_{\{g(x)=1\}}\eta_0(x) + \mathbb{I}_{\{g(x)=0\}}\eta_1(x) = 1 - \mathbb{I}_{\{x\in D_0\}}\eta_0(x) - \mathbb{I}_{\{x\in D_1\}}\eta_1(x)$$

Olyan g minimalizálja, ami $\forall x$-et a nagyobb $\eta_j(x)$-hez tartozó D_j-be sorol

BAYES-DÖNTÉS

- Legyen $\{D_j^*\}$ olyan, hogy $\forall x$

$$x \in D_j^* \Leftrightarrow (\eta_j(x) > \eta_{1-j}(x) \text{ v. } j = 0, \eta_0(x) = \eta_1(x))$$

g^* az X ismeretében valószínűbb j-t választja.

$$x \in D_j^* \Rightarrow \eta_j(x) = \max(\eta_0(x), \eta_1(x)).$$

DEFINITION

A fenti (D_0^*, D_1^*)-hoz tartozó g^* (azaz $g^*(x) = j \Leftrightarrow x \in D_j^*$) *Bayes-döntés* v. *maximum a posteriori döntés.*

THEOREM

A Bayes-döntés $\forall x$-re minimalizálja $r(g, x)$-et és így optimális. A minimum értéke $r(g^*, x) = \min(\eta_0(x), \eta_1(x))$.

g^* (optimális) globális kockázata: *Bayes-kockázat*/*Bayes-hiba*

$$R(g^*) = \mathbb{E}\left[\min(\eta_0(X), \eta_1(X))\right] = \mathbb{E}\left[\min(\eta_1(X), 1 - \eta_1(X))\right].$$

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$$x \in D_j^* \Leftrightarrow (\eta_j(x) > \eta_{1-j}(x) \text{ v. } j = 0, \eta_0(x) = \eta_1(x))$$

g^* az X ismeretében valószínűbb j-t választja.

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A fenti (D_0^*, D_1^*)-hoz tartozó g^* (azaz $g^*(x) = j \Leftrightarrow x \in D_j^*$) *Bayes-döntés* v. *maximum a posteriori döntés*.

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A Bayes-döntés $\forall x$-re minimalizálja $r(g, x)$-et és így optimális. A minimum értéke $r(g^*, x) = \min(\eta_0(x), \eta_1(x))$.

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OUTLINE

BAYES-DÖNTÉS KÖZELÍTÉSE

- $\{\eta_i(x)\}$ sokszor ismeretlen.
- Tfh. η_i-t becsülhető lehet valamely $\tilde{\eta}_i : \mathcal{X} \to [0, 1]$-vel.
- Bayes-döntés: $(\eta_0, \eta_1) \Rightarrow g^*$. Analógia: $(\tilde{\eta}_0, \tilde{\eta}_1) \Rightarrow \tilde{g}$:

$$\tilde{g}(x) = j \Rightarrow \tilde{\eta}_j(x) = \max(\tilde{\eta}_0(x), \tilde{\eta}_1(x)) \quad (1)$$

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- Várakozás: $\tilde{\eta}_i$-k jó becslések $\Rightarrow$ $\tilde{g}$ hibája $\approx$ g^* hibája (mindig $\geq$). Kockázatuk különbsége $\leq$ az $\tilde{\eta}_i$-k becslési hibája:

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BAYES-DÖNTÉS KÖZELÍTÉSE 2

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$$r(\tilde{g}, x) - r(g^*, x) \le \mathbb{I}_{\{\tilde{g}(x) \neq g^*(x)\}} \sum_{i \in \{0,1\}} |\tilde{\eta}_i(x) - \eta_i(x)|$$

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- Ha $\forall x \in \mathcal{X}$-re $\tilde{\eta}_0(x) = 1 - \tilde{\eta}_1(x) \Rightarrow r(\tilde{g}, x) - r(g^*, x) \le 2\mathbb{I}_{\{\tilde{g}(x) \neq g^*(x)\}} |\tilde{\eta}_1(x) - \eta_1(x)|$ és $R(\tilde{g}) - R(g^*) \le$

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OSZTÁLYOZÁS MINTÁKBÓL

Tfh. $\{\eta_i(x)\}$ ismeretlen, de adott n db i.i.d. minta: $(X_1, Y_1),\ldots,(X_n, Y_n) \sim (X, Y)$. Ezek alapján közelítjük g^*-ot. $\Rightarrow$ *Osztályozás* (v. *alakfelismerés*, néha *(felügyelt) tanulás*).
Legyen $|\mathcal{X}| < \infty$, $r_i(x) = \mathbb{P}(X = x, Y = i)$ $(x \in \mathcal{X}, i \in \{0, 1\})$ és $\tilde{r}_{in}(x)$ az $r_i(x)$ becslése az n mintából. Mivel $\eta_i(x) = r_i(x)/\mathbb{P}(X = x)$, legyen $\eta_i(x)$ közelítése

$$\tilde{\eta}_{in}(x) = \frac{\tilde{r}_{in}(x)}{\mathbb{P}(X = x)}, \qquad i \in \{0, 1\}$$

és $\tilde{g}_n$ egy hozzájuk rendelt döntésfüggvény. ($\mathbb{P}(X = x)$-k ismeretlenek $\Rightarrow$ $\tilde{\eta}_{in}(x)$-k nem meghatározhatók, de $\tilde{g}_n$-hez nem is kell, mert $\tilde{g}_n(x) = \operatorname{argmax}_i \tilde{\eta}_{in}(x) = \operatorname{argmax}_i \tilde{r}_{in}(x)$.)

$$R(\tilde{g}_n) - R(g^*) \leq \mathbb{E}\left[\sum_{i\in\{0,1\}} |\tilde{\eta}_{in}(X) - \eta_i(X)|\right] = \sum_{x\in\mathcal{X}} \sum_{i\in\{0,1\}} |\tilde{r}_{in}(x) - r_i(x)|.$$

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- Legyen $\tilde{r}_{in}(x) = \frac{1}{n}\sum_{t=1}^{n} \mathbb{I}_{\{X_t=x, Y_t=i\}}$ rel. gyakoriság.
- Nagy számok erős tv.-e: $n \to \infty \Rightarrow \forall x \in \mathcal{X}, i \in \{0,1\}$, $\tilde{r}_{in}(x) \to r_i(x)$ m.m.
- $\Rightarrow$ Jobb oldal 0-hoz tart m.m. és $R(\tilde{g}_n) - R(g^*) \geq 0 \Rightarrow$ az is $\to 0$.
- Így $R(g^*)$-ot „majdnem biztosan” tetszőlegesen megközelítő hibájú $\{\tilde{g}_n\}$ sorozatot kaptunk
- $|\mathcal{X}| < \infty$ esetén az eljárás *erősen konzisztens*.

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Ha n összemérhető az (X, Y) pár effektív értékkészletével, akkor bár az egyes tagok kicsik, az összeg már nem lesz az, így $R(\tilde{g}_n)$ megbízhatatlanná válik.

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OUTLINE

BAYES-BECSLÉS

Spec. eset: $\mathcal{Y}, \bar{\mathcal{Y}} \subseteq \mathbb{R}$, többnyire végtelen, folytonos, C az Y és $g(X)$ valamely távolságát méri a; hibázás nagysága is számít. Ekkor g *becslésfüggvény*.

DEFINITIONS

Bayes-becslésnek nevezzük az optimális g^* becslésfüggvényeket, azaz amikre $R(g^*) = \min_g R(g)$.

Fennáll a következő elégséges feltétel:

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Ha egy g^* becslésfüggvény lokális kockázatára $r(g^*, x) = \min_{y \in \bar{\mathcal{Y}}} \mathbb{E}\left[C(Y, y) | X = x\right]$ m.m. $x \in \mathcal{X}$-re, akkor g^* Bayes-becslés.

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REGRESSZIÓBECSLÉS; MSE MINIMALIZÁLÁS

- Legyen $Y \in \mathbb{R}$ véges szórású és $C(y, y') = C_2(y, y') = (y - y')^2$.
- $\Rightarrow$ keressük g^*-t, amire $R(g^*) = \mathbb{E}\left[(g^*(X) - Y)^2\right]$ min.
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THEOREM (STEINER)

$C = C_2$ esetén $\forall g$-re $r(g, x) = r(\mu, x) + (\mu(x) - g(x))^2$ $(\forall x \in \mathcal{X})$ és $R(g) = R(\mu) + \mathbb{E}\left[(\mu(X) - g(X))^2\right]$. Így $g^*(X) = \mu(X)$ m.m., azaz a Bayes-becslés a regressziós függvény. [Györfi et al., 2002, p. 2]

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$C = C_2$ esetén $\forall g$-re $r(g, x) = r(\mu, x) + (\mu(x) - g(x))^2$ $(\forall x \in \mathcal{X})$ és $R(g) = R(\mu) + \mathbb{E}\left[(\mu(X) - g(X))^2\right]$. Így $g^*(X) = \mu(X)$ m.m., azaz a Bayes-becslés a regressziós függvény. [Györfi et al., 2002, p. 2]

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- Legyen $Y \in \mathbb{R}$ véges szórású és $C(y, y') = C_2(y, y') = (y - y')^2$.
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OUTLINE

Active Learning

- Regression:
  - Goal: Learn f
  - Data: $(X_1, Y_1), \ldots, (X_n, Y_n)$, where

$$Y_t = f(X_t) + Z_t,$$
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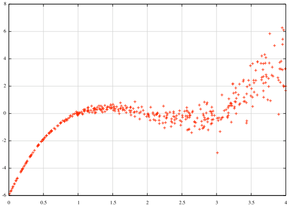

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"Active learning" — more efficient than passive learning for some problem classes

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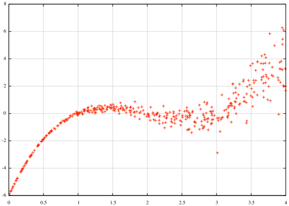

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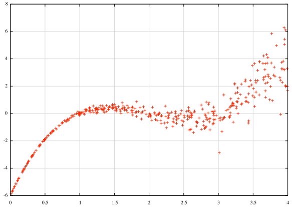

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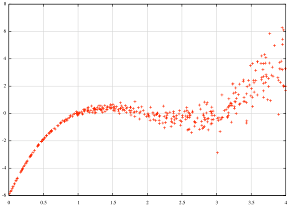

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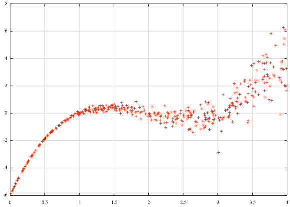

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OPTIMAL EXPERIMENT DESIGN/PLANNING/ALLOCATION

- E.g. [Fedorov, 1972]: mostly homoscedastic Gaussian noise: $\mathrm{Var}[Y|X=x] = \mathrm{Var}[Z|X=x] \equiv \sigma^2 = \mathrm{const}$
- E.g.: Generalized linear model: μ, $\phi(X) \in \mathbb{R}^d$ basis func.

$$Y = f(X) + Z = \mu^T \phi(X) + Z$$

  X has finite domain $\mathcal{X} = \{x_1, \ldots, x_K\}$, w.l.o.g. $x_k = k$. Now $\phi = [\phi(1) \cdots \phi(K)] \sim d \times k$ matrix $\Rightarrow$ optimization problem: given $\mathcal{X}$, how many times should we choose k to minimize some loss
- $K > d$: hard and interesting
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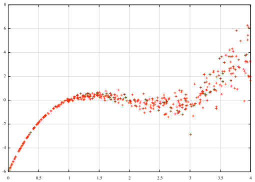
HETEROSCEDASTICITY

What if Z is heteroscedastic (non-Gaussian), i.e.,

$$\mathrm{Var}\left[Z|X=x\right]=\sigma^2(x),$$

and $\sigma^2(x)$ is unknown?



OUTLINE

Model & Protocol of Learning

- Given $(\mathcal{D}_k)_{1 \le k \le K}$ distributions with means $(\mu_k)_{1 \le k \le K}$ supported in $[0, 1]$.

Protocol of learning

1. Request sample from option/arm $X_t \in \{1, \dots, K\}$
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Weighted Loss

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Loss of $\hat{\mu}_{kn}$ after seeing n samples: $L_{kn} = \mathbb{E}\left[(\hat{\mu}_{kn} - \mu_k)^2\right]$. If k is chosen $\sim$ some a given $p_1, \ldots, p_K$, $p_k \geq 0$, $(\sum_k p_k = 1)$,

$$L_n^{(p)} = \sum_k p_k L_{kn}.$$

Examples

- Treatment efficiency testing
- Quality control
- ... whenever experiment design makes sense
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  - i.i.d. samples from Y on $\mathcal{S}_k$.
- $\mu_k \stackrel{\text{def}}{=} \mathbb{E}[Y|\mathcal{S}_k]$. $\Rightarrow$ $\mu = \sum_k w_k \mu_k$.
- **Goal**: Sample $Y|\mathcal{S}_k$ and contruct $\hat{\mu}_{kn}$ s.t. the loss of $\hat{\mu}_n = \sum_k w_k \hat{\mu}_{kn}$, $L_n = \mathbb{E}\left[(\hat{\mu}_n - \mu)^2\right] \to \min!$
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- Goal: Sample $Y|\mathcal{S}_k$ and contruct $\hat{\mu}_{kn}$ s.t. the loss of $\hat{\mu}_n = \sum_k w_k \hat{\mu}_{kn}$, $L_n = \mathbb{E}\left[(\hat{\mu}_n - \mu)^2\right] \to$ min!
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- If correlation terms vanishes (e.g., deterministic allocation): $L_n = \sum_k w_k^2 L_{kn} = \sum_k p_k L_{kn} \Rightarrow$ "weighted loss".

Examples

Running polls adaptively

ADAPTIVE STRATIFIED SAMPLING

- Problem: Estimate $\mu = \mathbb{E}[Y]$.
- Assumptions: We are given
  - strata $\mathcal{S}_1, \ldots, \mathcal{S}_K$: $\cup_k^* \mathcal{S}_k = \Omega$ with weights $w_k = \mathbb{P}(\mathcal{S}_k) > 0$.
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EXAMPLES

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MAX-LOSS

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Unknown $p \rightarrow$ pessimistic (worst-case) approach. Loss after seeing n samples:

$$L_n = \max_p L_n^{(p)} = \max_p \sum_{k=1}^{K} p_k L_{kn} = \max_{1 \leq k \leq K} L_{kn}, \quad \text{where}$$

$L_{kn} = \mathbb{E}\left[(\hat{\mu}_{kn} - \mu_k)^2\right]$.
Estimates of all options are uniformly accurate.

EXAMPLES

- Treatment efficiency testing
- Quality control
- ... whenever experiment design makes sense

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SAMPLES AND ESTIMATE FOR AN OPTION

- $\forall k$ learner sequentially decides the frequency of $\{X_t = k\}$
- $T_{kn} \stackrel{\text{def}}{=} \sum_{t=1}^{n} \mathbb{I}_{\{X_t=k\}} \geq 0$: number of samples allocated to arm k till time n

$$\sum_k T_{kn} = n.$$

- $Y_{kt} \stackrel{\text{def}}{=}$ sample requested the t^{th} times from arm k
- μ_k=mean $\Rightarrow$ $\hat{\mu}_{kn} \stackrel{\text{def}}{=}$ sample average of arm k:

$$\hat{\mu}_{kn}(=\bar{Y}_{kn}) = \frac{1}{T_{kn}} \sum_{t=1}^{T_{kn}} Y_{kt}.$$

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Outline

OPTIMAL ALLOCATION (MAX-LOSS)

- Assume $\sigma_k^2 = \mathrm{Var}[Y_{kt}] = \mathrm{Var}[Y|X=k]$ are known (and > 0 now). Choose $\{T_{kn}\}_k$ in advance (not data-dependent).
- Loss of option k: variance of average of i.i.d. $\Rightarrow$

$$L_{kn} = \frac{\sigma_k^2}{T_{kn}}$$

- Goal: Minimize $L_n = \max_k L_{kn}$!
- Solution: Optimal choice must make all L_{kn} the same:

$$T_{kn}^* = n\frac{\sigma_k^2}{\Sigma^2} \stackrel{\text{def}}{=} n\lambda_k \qquad \text{(mod rounding)},$$

where $\Sigma^2 = \sum_k \sigma_k^2$, λ_k is *optimal allocation ratio* for arm k.
- $\Rightarrow$ To get $\{T_{kn}^*\}_k$ only $\{\sigma_k\}_k$ is needed
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- The optimal loss under the optimal allocation:

$$L_n^* = \frac{\Sigma^2}{n}.$$

- Loss of uniform allocation ($T_{kn} \equiv n/K$):

$$L_n^{(U)} = \max_k \frac{K\sigma_k^2}{n} = \frac{K \max_k \sigma_k^2}{n} \geq \frac{\Sigma^2}{n}$$

'$=$' $\Leftrightarrow$ homoscedastic case

- $L_n(\mathcal{A}) \stackrel{\text{def}}{=}$ loss of algorithm $\mathcal{A}$

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Explore or exploit?

Idea

- Estimate $\hat{\sigma}^2_{kt}$ for all k at time t: $\hat{\sigma}^2_{kt} = \frac{1}{T_{kt}} \sum_{i=1}^{T_{kt}} Y^2_{ki} - \hat{\mu}^2_{kt}$
- Sample arm $X_{t+1} \in \operatorname{argmax}_k \frac{\hat{\sigma}^2_{kt}}{T_{kt}}$! (based on earlier samples)

- Is this a good idea?
- Reminds to *multi-armed bandit problems* [Auer et al., 2002]. *Exploration-exploitation trade-off*! (But different performance criterion.)
- How to explore?
  - Upper confidence bounds
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- Is this a good idea?
- Reminds to *multi-armed bandit problems* [Auer et al., 2002]. *Exploration-exploitation trade-off*! (But different performance criterion.)
- How to explore?
    - Upper confidence bounds
    - ϵ-greedy
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EXPLORE OR EXPLOIT?

IDEA

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OUTLINE

ALGORITHM GAFS-MAX

Greedy Allocation with Forced Selection ⇒ GAFS

Algorithm GAFS-MAX(α)

1. First K trials: choose each arm once, initialize $\forall k: T_{kK} = 1, \hat{\sigma}_{kK} = 0$
2. At time $t = K + 1, \ldots, n$ do:
3. $U_{t-1} \in \mathrm{argmin}_{1 \le k \le K} T_{k,t-1}$
4. Let

$$X_t = \begin{cases} U_{t-1}, & \text{if } T_{U_{t-1},t-1} < \alpha\sqrt{t-1} + 1 \\ \in \mathrm{argmax}_{1 \le k \le K} \frac{\hat{\sigma}^2_{k,t-1}}{T_{k,t-1}}, & \text{otherwise} \end{cases}$$

5. Choose option X_t, $T_{kt} := T_{k,t-1} + \mathbb{I}_{\{X_t = k\}}$
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OUTLINE

UPPER BOUND GAFS-MAX

THEOREM

Let $\lambda_{\min} = \min_{k:\lambda_k>0} \lambda_k$. *Then there is* $C = \text{poly}(1/\lambda_{\min})$, *s.t. for any* n

$$R_n(\mathcal{A}_{\text{GAFS-MAX}}) \leq Cn^{-3/2}\sqrt{\ln n}.$$

Proof – preparations

Observations

With high probability:

- **Claim 1**: If $T_{kn} \geq f(n)$ then

$$\hat{\lambda}_{kn} - \lambda_k = O\left(\sqrt{\frac{\ln n}{f(n)}}\right)$$

- **Claim 2**: If $\hat{\lambda}_{kn} - \lambda_k = O(\sqrt{\ln n / f(n)})$ then

$$\frac{T_{kn}}{n} - \lambda_k = O\left(\sqrt{\frac{\ln n}{f(n)}}\right).$$

PROOF – PREPARATIONS

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PROOF (SKETCH)

PROOF.

- Forcing $\Rightarrow T_{kn} \geq \sqrt{n}$
- Use Claim 1 & 2 with $f(n) = \sqrt{n} \Rightarrow$

$$\frac{T_{kn}}{n} - \lambda_k = O\left(\sqrt{\frac{\ln n}{n^{1/2}}}\right).$$

- $\Rightarrow T_{kn} \geq \frac{\lambda_k}{2} n$ if n is large enough
- Use Claim 1 & 2 with $f(n) = cn \Rightarrow$

$$\frac{T_{kn}}{n} - \lambda_k = O\left(\sqrt{\frac{\ln n}{n}}\right).$$

- Wald's second identity $\Rightarrow L_{kn} - L_n^* = O(n^{-3/2}\sqrt{\ln n})$.

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OUTLINE

ALGORITHM GFSP-MAX

Greedy Forced Selection with Phases $\Rightarrow$ GFSP

Phase m: times $\{t_m - K + 1, \ldots, t_m\} \cup \{t_m + 1, \ldots, t_m + m\}$

Algorithm GFSP-MAX

1. First $2K$ trials: choose each arm twice, initialize $\forall k : T_{k,2K} = 2, \hat{\sigma}_{k,2K}$
2. In phase m, $t_m \stackrel{\text{def}}{=} m(2K + m - 1)/2 + 2K$ and do:
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2. In phase m, $t_m \stackrel{\text{def}}{=} m(2K + m - 1)/2 + 2K$ and do:
3. At times $t_m - K + 1, \dots, t_m$: choose each arm once
4. $\hat{\Sigma}^2 \stackrel{\text{def}}{=} \sum_k \hat{\sigma}^2_{k,t_m - K}$, $\hat{\lambda}_{m,k} \stackrel{\text{def}}{=} \hat{\sigma}^2_{k,t_m - K}/\hat{\Sigma}^2$
5. At times $t_m + 1, \dots, t_m + m$: choose arm k $m\hat{\lambda}_{k,m}$ times (+rounding!)
6. $T_{k,t_m + m} := T_{k,t_{m-1}+m-1} + m\hat{\lambda}_{k,m} + 1$
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8. Update estimate $\hat{\mu}_{k,t_m+m}$ and $\hat{\sigma}^2_{k,t_m+m}$.

ALGORITHM GFSP-MAX

Greedy Forced Selection with Phases $\Rightarrow$ GFSP
Phase m: times $\{t_m - K + 1, \ldots, t_m\} \cup \{t_m + 1, \ldots, t_m + m\}$

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OUTLINE

ANALYSYS GFSP-MAX

- Observe: After phase m:
  - each arm is chosen $\geq m+2$ times,
  - number of all trials: $2K + Km + (1+2+\ldots+m) \approx m^2/2$.
- $\Rightarrow$ by time t each arm is chosen at least approximately $\sqrt{2t}$ times.
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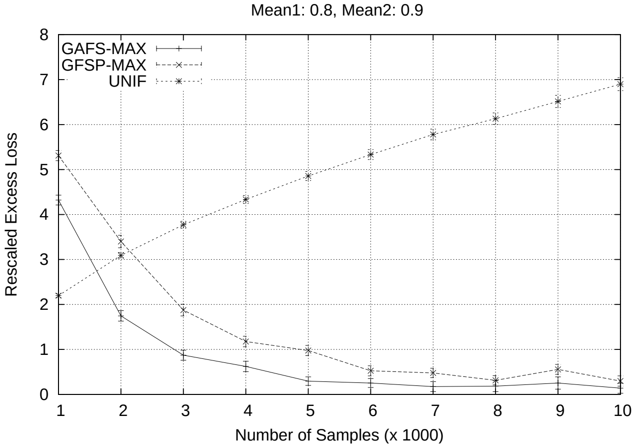
BERNOULLI OPTIONS, $K = 2$



The rescaled excess loss, $n^{3/2} R_n$, against n

OUTLINE

ALGORITHM UCB-CH-AS-MAX

Upper Confidence Bounds Chernoff-Hoeffding Allocation Strategy ⇒ UCB-CH-AS

Algorithm UCB-CH-AS-MAX(δ)

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4. Let $X_t \in \operatorname{argmax}_{1 \le k \le K} B_{k,t}$,
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OUTLINE

UPPER BOUND UCB-CH-AS-MAX

THEOREM

Let $\lambda_{\min} = \min_{k:\lambda_k>0} \lambda_k$*. Then there is* $C = \mathrm{poly}(1/\lambda_{\min}, 1/\Sigma)$*, s.t. for any* n*, if it is known in advance, and UCB-CH-AS-MAX runs with* $\delta = n^{-5/2}$*, then*

$$R_n(\mathcal{A}_{\mathrm{UCB-CH-AS-MAX}}) \leq Cn^{-3/2}\sqrt{\ln n}.$$

PROOF (SKETCH)

For $\delta > 0$, let $\xi = \xi_{K,n}(\delta)$ be event

$$\bigcap_{\substack{1 \le k \le K \\ 1 \le t \le n}} \left\{ \left| \frac{1}{t} \sum_{i=1}^{t} Y_{ki}^2 - \left(\frac{1}{t} \sum_{i=1}^{t} Y_{ki} \right)^2 - \sigma_k^2 \right| \le 3 \sqrt{\frac{\ln(1/\delta)}{2t}} \right\}.$$

- Hoeffding: $\mathbb{P}(\xi) \ge 1 - 4Kn\delta$.
- There is $C = \mathrm{poly}(1/\lambda_{\min}, 1/\Sigma)$, s.t. if UCB-CH-AS-MAX runs with δ, then $\forall n$ on ξ, $\forall k$: $|T_{k,n} - T_{k,n}^*| \le C\sqrt{n \ln(1/\delta)}$.

CONSEQUENCE

With high probability: $|T_{k,n} - T_{k,n}^*| = O(\sqrt{n \ln(1/\delta)})$.

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Conclusions

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- ..Even if the rates are the same!
- Plug-in rules with forcing or UCB can work well

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  - Classification
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