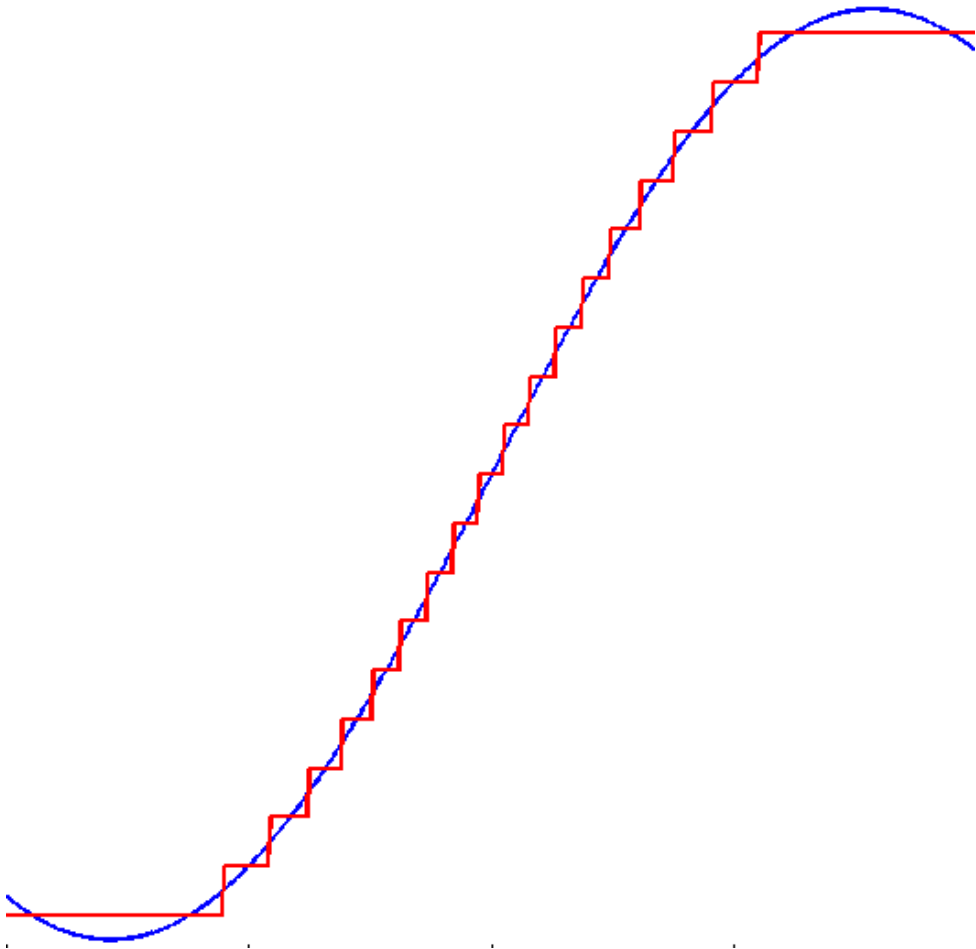


# ADC test data evaluation program for MATLAB

## User's manual



Version 5.0

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# 1 Introduction

## 1.1 Scope

This software is a MATLAB toolbox with a graphical user interface: a collection of functions that can be executed in MATLAB environment. The toolbox provides the implementation of the following algorithms used in ADC testing:

- Four parameter sine wave fitting method described in standard IEEE-1241
- FFT test
- Histogram test using sinusoidal excitation signal
- Maximum likelihood (ML) estimation of signal and ADC parameters (with sinusoidal excitation signal)
- Approximate maximum likelihood (AML) estimation signal and ADC parameters (with sinusoidal excitation signal)

Furthermore, the toolbox contains a simple tool to generate simulated measurement data: this possibility is useful to test and compare the fitting algorithms.

## 1.2 What's new?

Version 5.0 is a major technical update. The toolbox has been reformulated to fit MATLAB's App Designer development environment, instead of the obsolete Matlab GUIDE. It contains minor changes in the user interface and bug fixes.

Compared to the latest release of ADCTest toolbox (ver. 4.5), version 4.6 contains minor bug fixes and unified notation regarding approximate maximum likelihood (AML) estimation of quantizer and signal parameters.

Compared to the earlier version of ADCTest toolbox (ver. 4.4) the main new feature of version 4.5 is the approximate maximum likelihood (AML) estimation of quantizer and excitation signal parameters. This method parameterizes the integral nonlinearity of the quantizer under test and optimizes the ML cost function with respect to the parameters of the excitation signal and the parameters of the quantizer nonlinearity as well. This new feature is detailed in subsection 5.5

Compared to the earlier versions of ADCTest toolbox (ver. 4.1, 4.2, 4.3), the most important new feature is the extended cost function evaluator (`EvaluateCFExtended.m`) in maximum likelihood estimation of ADC and signal parameters. This new function calculates the entire gradient (size:  $2^b + 4$ ) and Hessian matrix of the ML cost function (size:  $(2^b + 4) \cdot (2^b + 4)$ ). The Hessian matrix provides the full Fisher information regarding the estimation of code transition levels and signal parameters. This way the toolbox calculates the Cramér-Rao Lower Bound for the covariance of all estimators. The most important values of the CRLB can be displayed in the GUI at the comparison of the results of ML and LS estimation using the 'Show CRLB' pushbutton (see subsection 5.4.3 and figure 15).

Compared to the earlier versions of ADCTest (ver 3.1 3.2 3.3) the most important new feature is the implementation of maximum likelihood (ML) parameter estimation to sinusoidal excitation signal [2]. The estimation of the converter's static transfer characteristic via histogram test is

also a new possibility in this toolbox. This method is based on [3]. The software has been expanded with FFT test function. The simulated measurement generator tool is a side-effect of the development process.

## 2 Installation

1. Download the latest version of `adctest` toolbox from the project site ([7]) in a zip file.
2. Extract the zip file to a folder used to store MATLAB toolboxes e. g. `C:\Program Files\MATLAB\toolbox`
3. Start MATLAB.
4. Add the directory (where you have installed the toolbox) to MATLAB search path using the *File->Set Path* menu or the *addpath* command.
5. Type *adctest* to the MATLAB command line to launch the GUI.

### 3 Getting started

Once the toolbox is launched, the main window appears (see Figure 1).

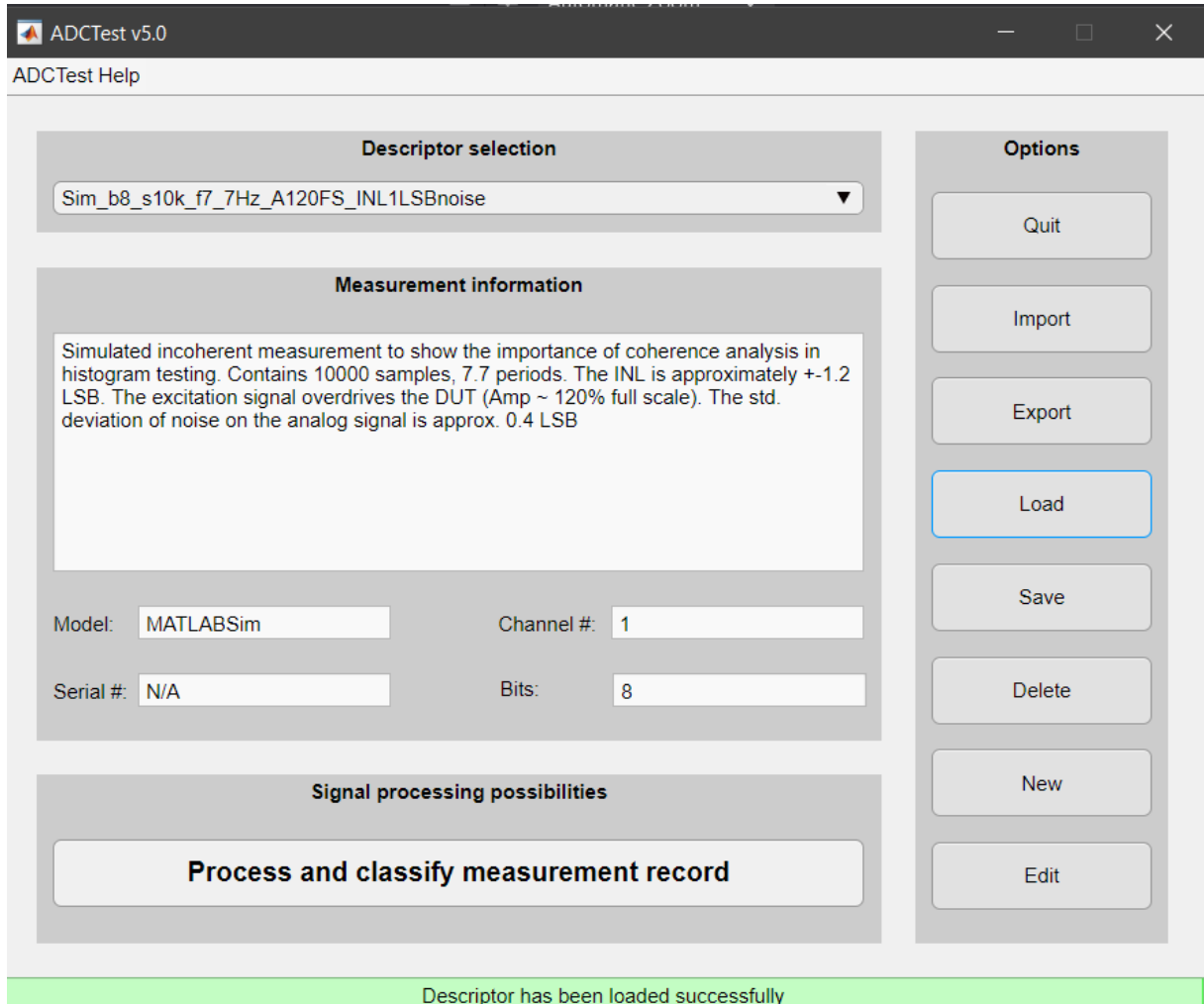


Figure 1: Main window of the toolbox

On the right side of the window, the following data handling options are available:

- Importing multiple measurement descriptors from workspace
- Exporting multiple measurement descriptors to workspace
- Loading a single measurement descriptor
- Saving the current measurement descriptor
- Deleting the current measurement descriptor
- Creating a new measurement descriptor
- Editing an existing measurement descriptor

At the bottom of the window the "Process and classify measurement record" pushbutton appears. Using this pushbutton, the user can reach the classification dialog box. This window shows the classification results with respect to the following test methods:

- Four parameter sine wave fit in LS sense
- FFT test of the ADC
- Histogram test of the ADC using sinusoidal excitation signal
- Maximum likelihood estimation of signal and ADC parameters
- Approximate maximum likelihood (AML) estimation of signal and ADC parameters

The result of classification can be

- Appropriate
- Appropriate with restrictions
- Inappropriate

Classification window appears in Figure 2

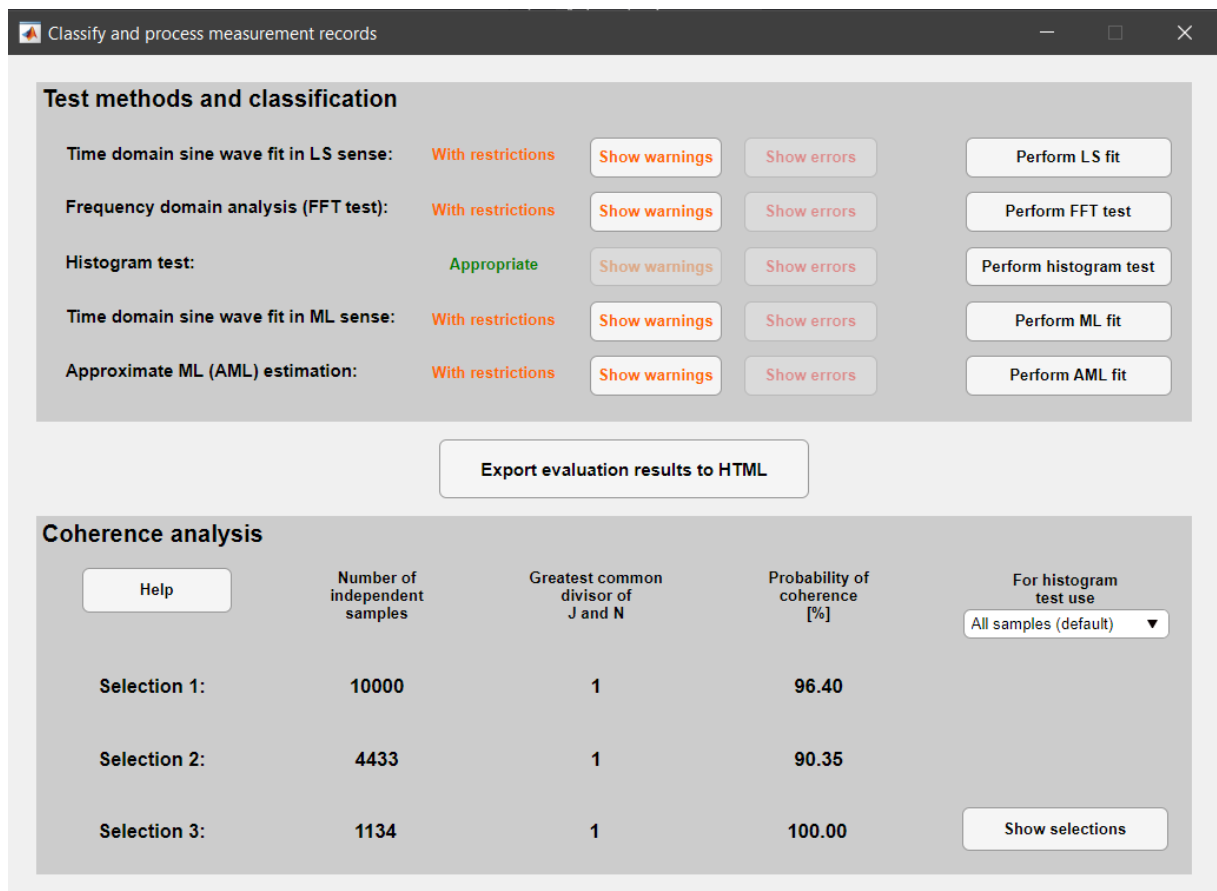


Figure 2: Classification of measurement record

If a measurement record is "Appropriate" or "Appropriate with restrictions", to process in a certain way, the data can be evaluated using the active pushbuttons ("Perform ...") on the right side. Possible warnings can be displayed using the "Show warnings" pushbutton. If the measurement record is inappropriate to process, an error message can be read using the "Show errors" pushbutton. The evaluation results can be exported to an HTML document using the "Export results to HTML" pushbutton. A typical HTML report appears in Figure 3

Measurement Evaluation Results		
generated by <a href="#">ADCTest toolbox</a> on 07-Dec-2022		
<b>Device Under Test</b>		
Model:	MATLABSim	
Serial number:	MATLABSim	
Channel:	1	
Number of bits:	8	
<b>Evaluation results</b>		
• Sine wave fit in time domain		
<b>Dynamic property</b>	<b>Using LS fit</b>	<b>Using ML fit</b>
Effective Number of Bits (ENOB):	3.46	N/A
Signal to Noise and Distortion Ratio (SINAD):	23.73 dB	N/A
• Frequency domain analysis (FFT test)		
<b>Property</b>	<b>Value</b>	
Spurious-Free Dynamic Range (SFRD) relative to carrier (dBc):	23.15 dB	
Spurious-Free Dynamic Range (SFRD) relative to the full scale (dBFS):	22.31 dB	
• Estimation of static transfer characteristic (histogram test)		
<b>Property</b>	<b>Minimum [LSB]</b>	<b>Maximum [LSB]</b>
Integral nonlinearity (INL):	-2.12	2.11

Figure 3: Evaluation results exported to a HTML report

The classification window also provides a feature called coherence analysis. The toolbox examines the measurement record and estimates the frequency via least squares fitting in the frequency domain (using Blackman-Harris windows). Based on this frequency estimator the algorithm suggests 3 selections of the measurement record that are coherent with a large probability. In this case, coherence means to fulfill the Carbone-Chiorboli conditions [5] and the probability of coherence is calculated using the confidence interval of the frequency estimator and the strict limits provided by the Carbone-Chiorboli condition. This coherence analysis is based on [6]. The selections are plotted in a dialog box, as well (see Figure 4).



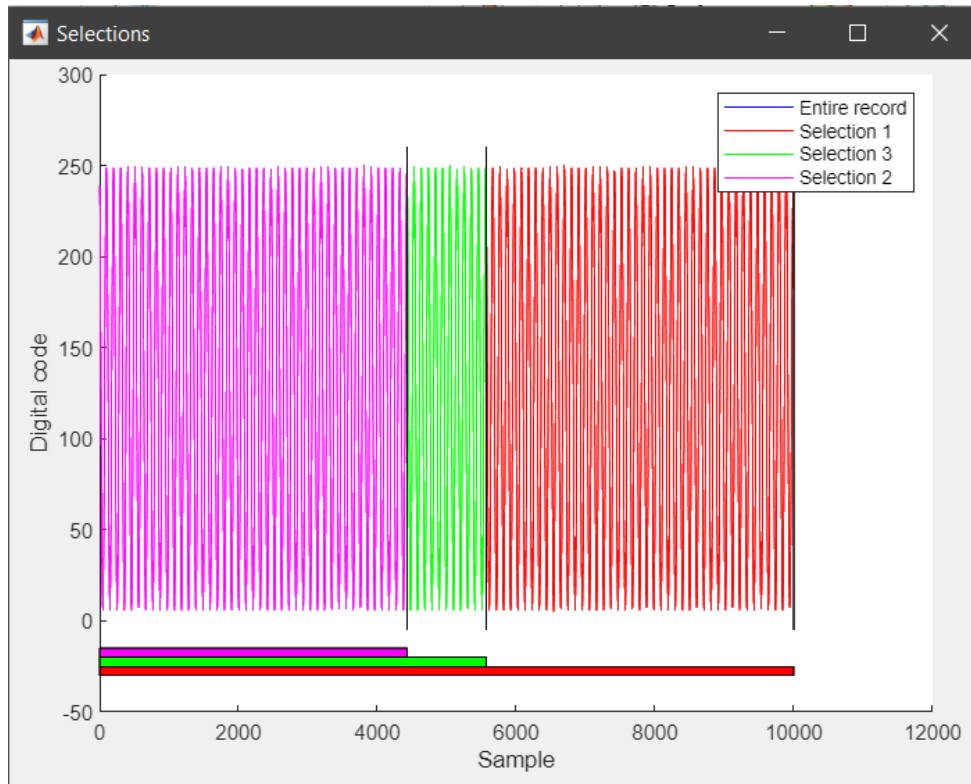


Figure 4: Graphical display of the coherent parts of the measurement record

## 4 Description of measurements

The description of measurements is realized using XML files since version 3.3. The proper schema definition of these XML files is available on the project's web site [7]. These descriptors have mandatory and optional fields. The descriptors must contain

- The name of the measurement: a short but informative title. Usually used to be the filename of the XML
- The model of the ADC under test
- The serial number of the ADC under test
- The resolution (bit number) of the ADC under test
- The measurement data: the record of the ADC's output (in digital codes).

The descriptor may also contain the following information:

- Comment: a more detailed description of measurement
- Channel: useful if the ADC under test is a multi-channel device
- Parameters: a feature to support test runs. In case of simulated measurement, the original signal parameters and the ADC's static transfer characteristic (INL vector) can be saved. Thus, the accuracy of parameter estimation can be calculated using the original and the fitted values.

### 4.1 Loading an existing measurement descriptor

This function is available from the main window, using the "Load" pushbutton. If the XML file is deficient, the load function returns error messages or warnings: errors, when the missing fields are mandatory, and warnings if the missing fields are optional.

### 4.2 Creating a new measurement descriptor

This function is available from the main window, using the "New" pushbutton. A dialog box appears (Figure 5), where all properties of the measurement descriptor can be set.

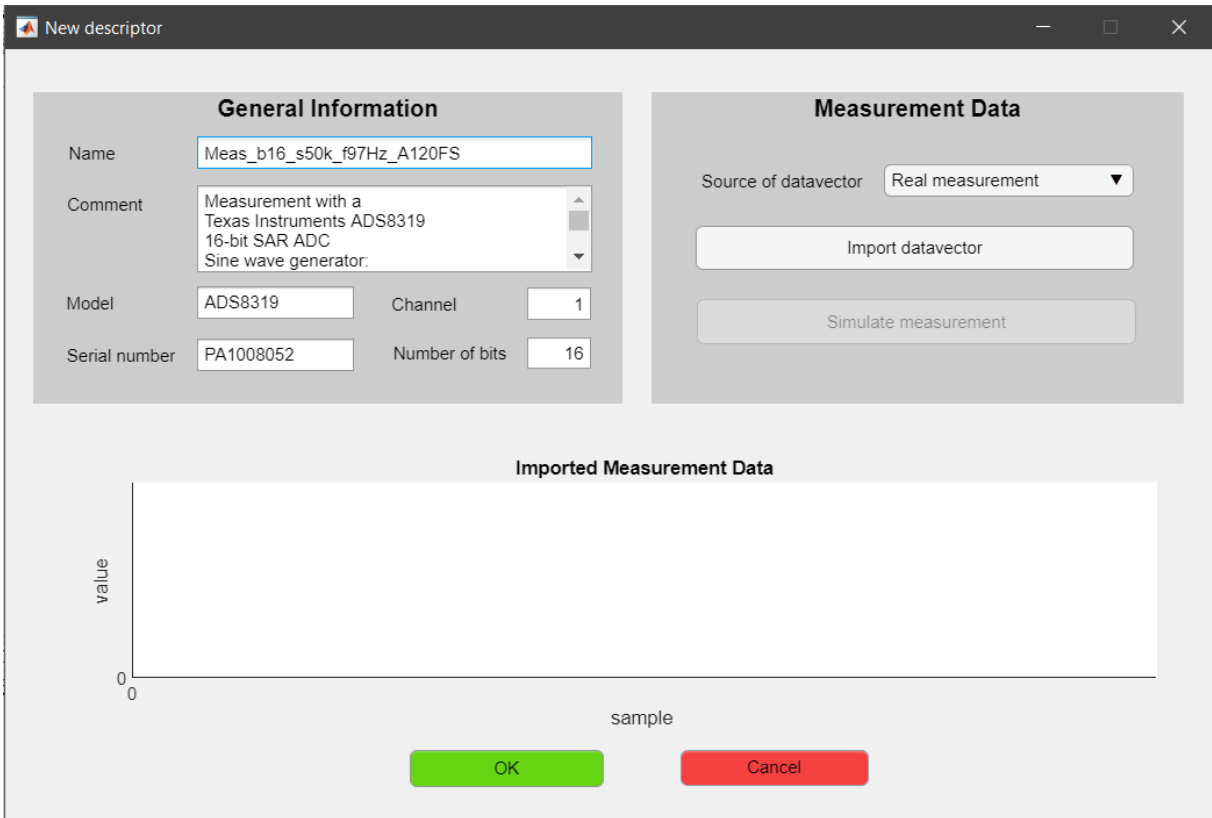


Figure 5: Add new descriptor dialog box

In case of a real measurement, the ADC output data vector can be loaded from the MATLAB workspace. In case of a simulated measurement, a simulation tool can be started (see Section 4.7).

### 4.3 Editing an existing measurement descriptor

This function is available from the main window, using the "Edit" pushbutton. The dialog box is similar to the dialog box "Add new descriptor", but the original properties and values of the descriptor are displayed in the appropriate fields of the window.

### 4.4 Batch import of measurement descriptors

This function is available from the main window, using the "Import" pushbutton. A variable from the MATLAB workspace that contains multiple descriptors can be imported into the toolbox. Note that no software component checks that your variable is appropriate: if the format of the data in this variable is not matching, strange errors can occur. Used to perform test runs.

### 4.5 Batch export of measurement descriptors

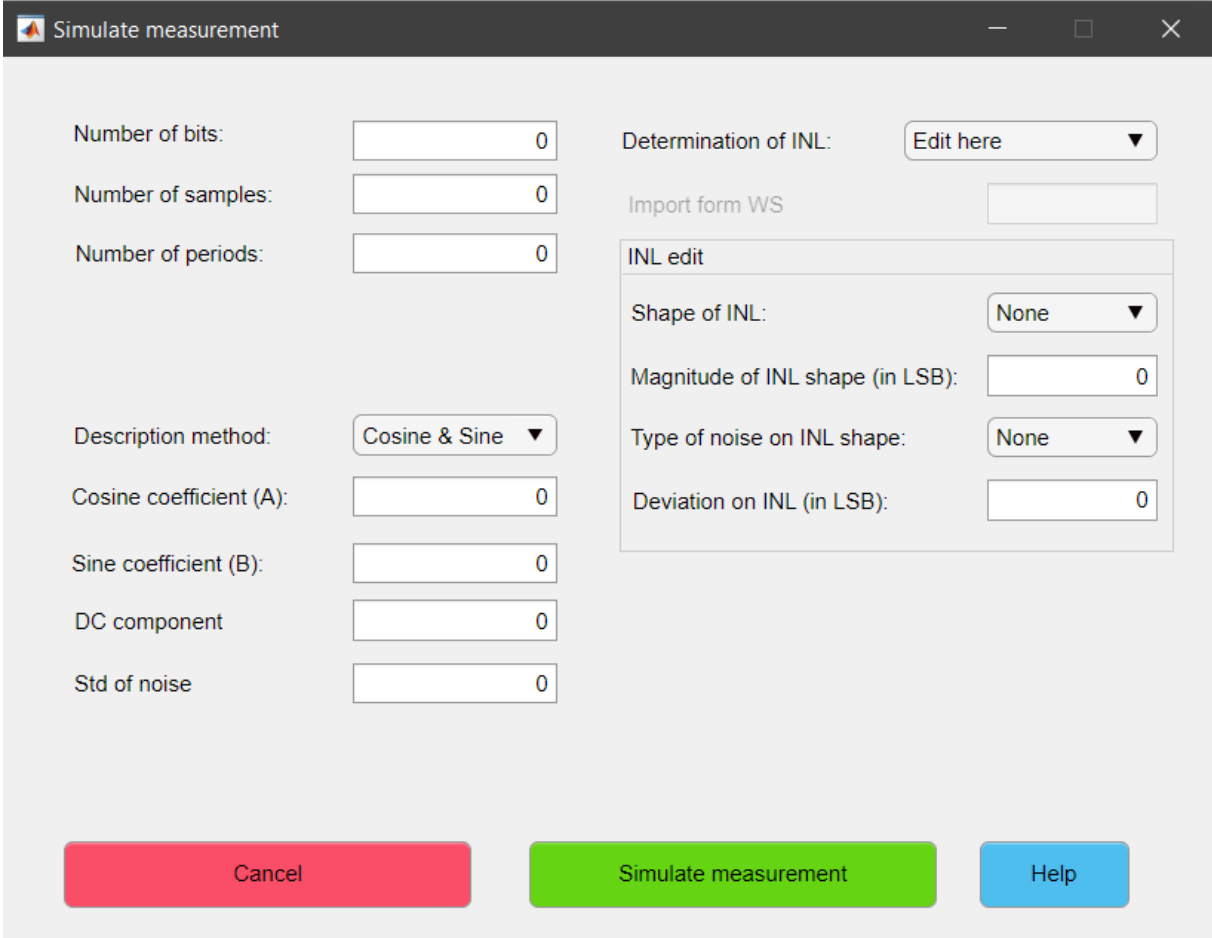
This function is available from the main window, using the "Export" pushbutton. All the measurement descriptors that are currently in the memory, will be copied to the workspace. The variable specified by the user is a cell array, each cell containing one measurement.

## 4.6 Saving a measurement descriptor

This function is available from the main window, using the "Save" pushbutton. The filename suggested by the save dialog box is the name of the descriptor. Keeping this file naming convention is recommended.

## 4.7 Creating a new simulated measurement descriptor

This function is available from the dialog box "Add new descriptor". If the "Source of the data vector" is changed from "Real measurement" to "Simulated measurement", a pushbutton "Simulate Measurement" appears. This button launches the simulation tool (Figure 6).



The dialog box "Simulate measurement" contains the following parameters:

- Number of bits: 0
- Number of samples: 0
- Number of periods: 0
- Description method: Cosine & Sine
- Cosine coefficient (A): 0
- Sine coefficient (B): 0
- DC component: 0
- Std of noise: 0
- Determination of INL: Edit here
- Import form WS: [empty field]
- INL edit section:
  - Shape of INL: None
  - Magnitude of INL shape (in LSB): 0
  - Type of noise on INL shape: None
  - Deviation on INL (in LSB): 0

Buttons: Cancel, Simulate measurement, Help

Figure 6: Simulate measurement dialog box

In this dialog box, both the excitation signal parameters and the ADC parameters can be set. The sine wave can be described in two ways: amplitude and initial phase, or cosine and sine coefficients. The nonlinearity can be imported from the workspace (e.g., the nonlinearity of a previously measured ADC), or can be assembled in this dialog box. The nonlinearity assembler uses two components: a component that determines the nature of the nonlinearity (e.g., a raised cosine (Hann) window), and a component that adds some noise to the INL vector. This way, the transition level values will not be so regular: the transfer characteristic will be more lifelike. Note that the "Deviation of noise on INL" means the standard deviation of noise in both cases: so in the case of normally distributed noise, as well as in the case of uniformly distributed noise.

## 5 Evaluation of measurements

### 5.1 Four parameters sine wave fitting test

To perform four parameter sine wave fit according to standard IEEE-1241-2010, some parameters of optimization must be set. Before starting the optimization, a dialog box appears (Figure 7).

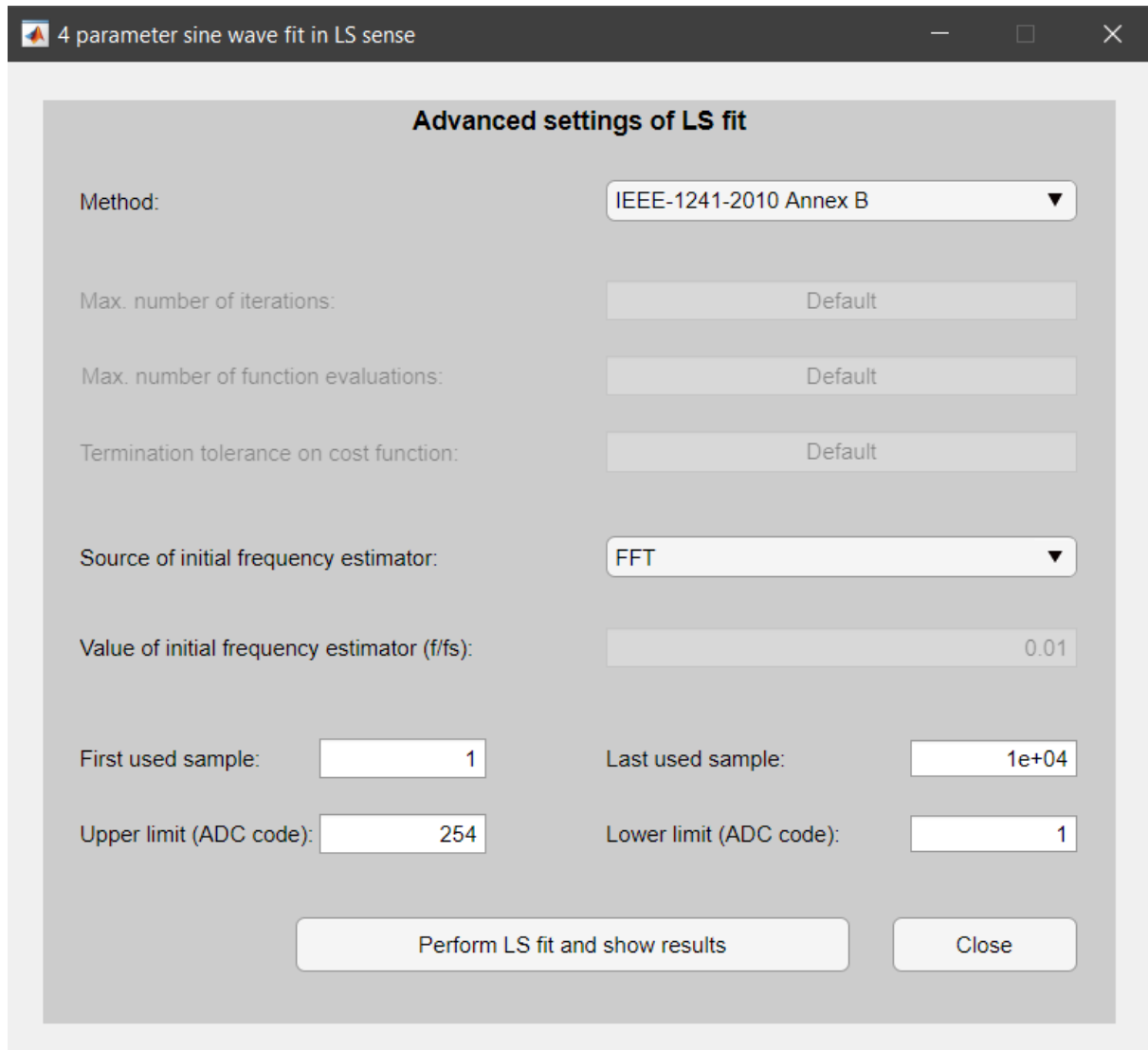


Figure 7: Settings window for four parameter LS fit

On the left side, the boundaries of the record can be set. Using "First sample of record" and "Last sample of record" the useless parts of the recording (like initial transients) can be cut. "Upper boundary" and "Lower boundary" determines the code range, where the samples are taken into consideration. This function is used to throw samples where the ADC is overdriven. Note that the default lower boundary is the lowest code of ADC + 1, and the default upper boundary is the highest code of ADC - 1.

On the right side, the parameters of optimization can be set. The following fitting methods are available:

- IEEE-1241-2010 Annex B: performs the method described in [1]
- LsqNonlin: calls the built-in nonlinear least squares solver function of MATLAB Optimization toolbox. This method is more time-consuming, but can be more robust in some cases.

In case of LsqNonlin, three other parameters of optimization can be set:

- Maximal number of iterations
- Maximal number of cost function evaluations
- Termination tolerance on cost function

The results of the sine wave fitting appear in the results window (Figure 8).

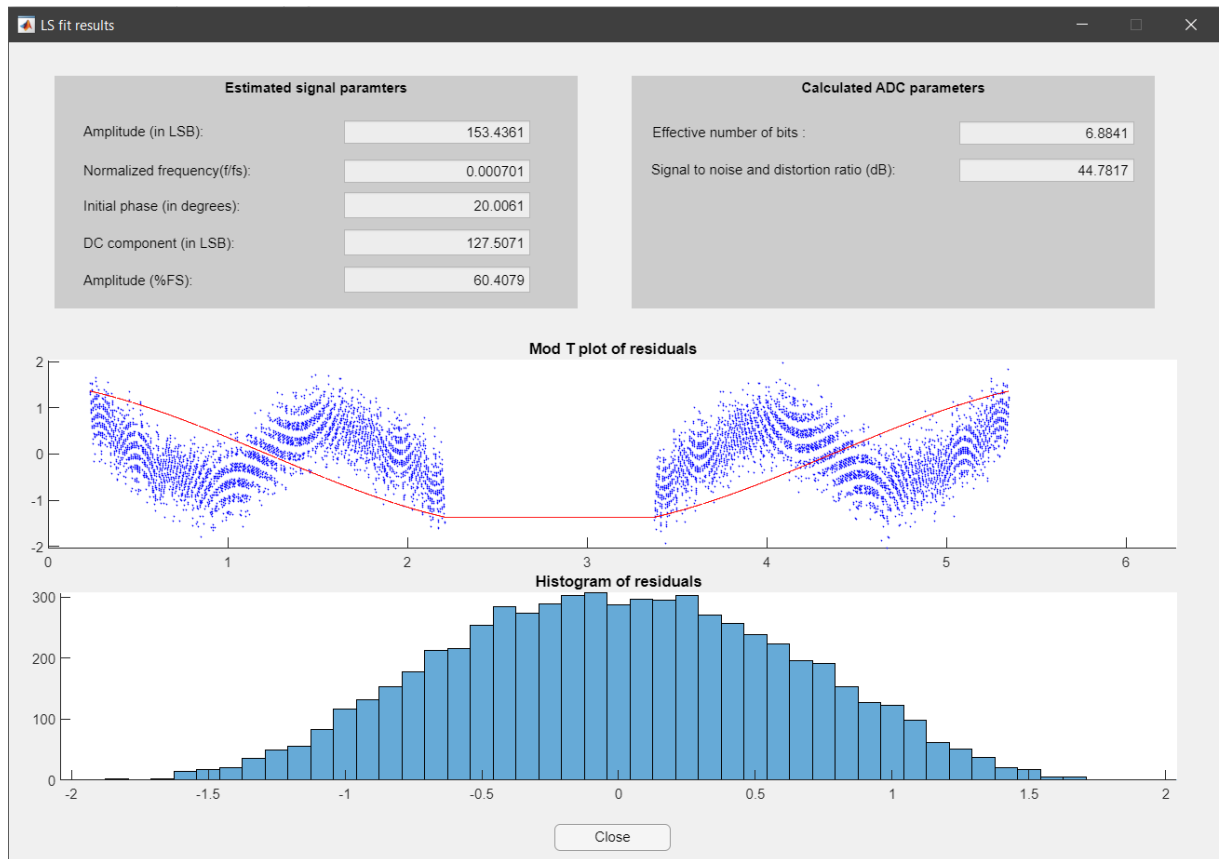


Figure 8: Results window for four parameter LS fit

This dialog box displays the estimated sine wave parameters (amplitude, frequency, initial phase, DC component), and the ADC parameters: effective number of bits (ENOB), and signal to noise and distortion ratio (SINAD). ADC parameters are calculated from the difference between the fitted sine wave and the measurement data, that is, from the residuals. The residuals are displayed in two ways:

- **Mod T plot:** the residuals are sorted and displayed according to their location in the phase space. The phase axis is displayed between 0 and  $2\pi$ . A reference sine wave is also plotted: this is a sine wave with the estimated phase of the excitation signal.
- **Histogram of residuals:** to observe statistical moments of the residuals.

## 5.2 FFT test: analysis in the frequency domain

The frequency domain analysis investigates the Discrete Fourier Transform (DFT) of the measurement record. DFT is performed using the Fast Fourier Transform (FFT) algorithm. Windowing in the time domain is necessary to suppress spectral leakage in case of incoherent sampling. The user can specify the type of window: the default is the *Blackman* window, but *Hann* and *3-term Blackman-Harris* windows can also be applied. FFT test can be performed without windowing, choosing the *None (rect)* option.

The results provide information about the amplitudes and frequencies of the harmonics of the signal (up to 5 harmonics) relative to the carrier (base harmonic) and relative to the full scale of the ADC. The most important quantity calculated this way is the **Spurious-free dynamic range (SFDR)**, which is provided relative to the carrier (*dBc*), and relative to the full scale (*dBFS*).

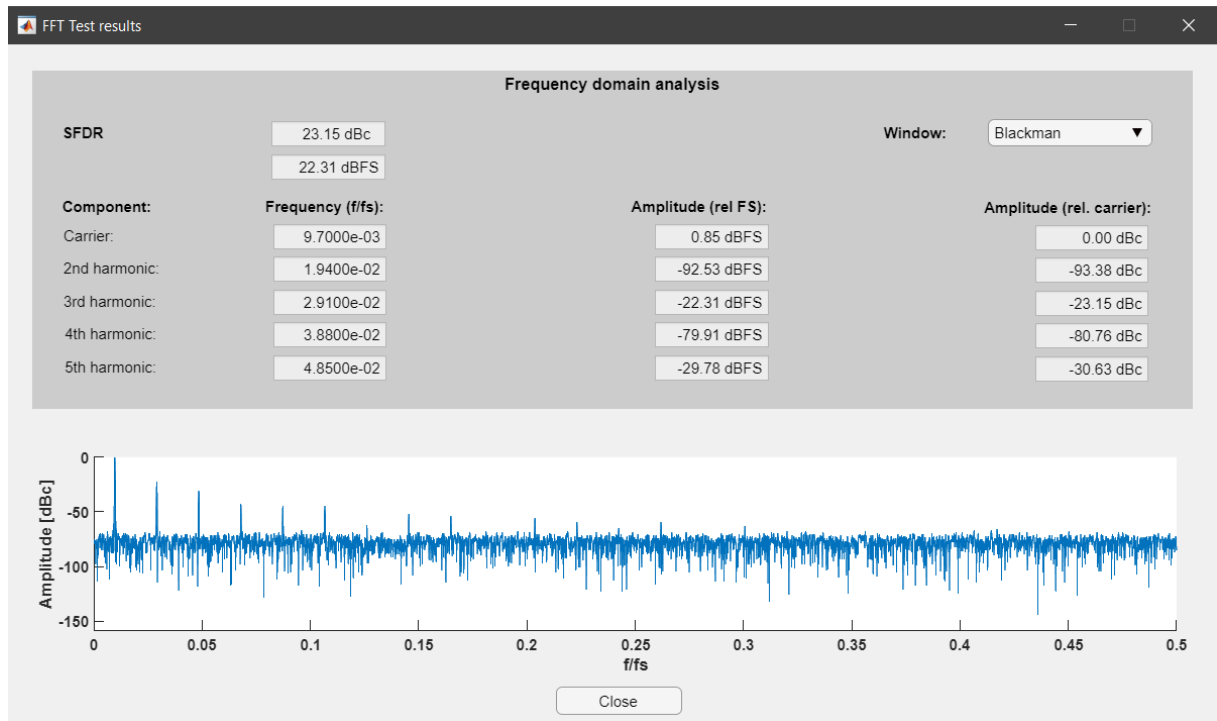


Figure 9: FFT test results window

### 5.3 Histogram test

Histogram test is the estimation of the static transfer characteristic of the ADC under test. The test uses the histogram of measured data and calculates the INL estimators described in [4]. A formula about the accuracy of these estimators is given in [3].

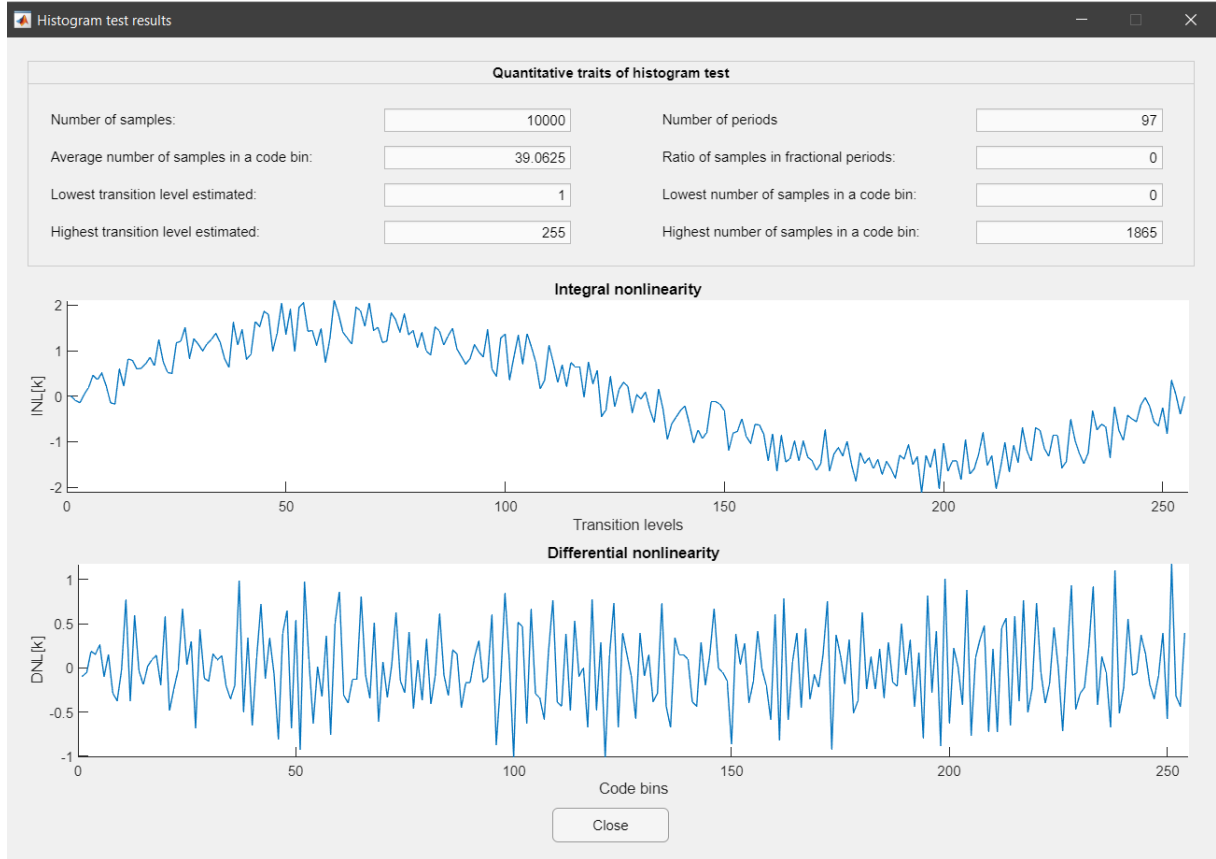


Figure 10: Result window of histogram testing

### 5.4 Maximum likelihood estimation of signal and ADC parameters

To perform maximum likelihood (ML) estimation of signal and ADC parameters, it is necessary to use the results of the histogram test, as well as the results of sine wave fitting in least squares (LS) sense. The histogram test provides the transition level estimators. Note that the transition level estimators do not change during the optimization, according to the problems described in [4]. Only the five signal parameters (four sine wave parameters, and the standard deviation of noise on the input) are optimized, keeping the transition levels constant. If the histogram is rough according to the low number of samples in a record, the transition level estimators can be inaccurate. The main source of inaccuracy of ML signal parameter estimators is the inaccuracy of transition levels. Before performing ML optimization, make sure that there is a sufficient number of samples in the record using the formula described in [3]. The LS sine wave fitting is used to get initial estimators of the sine wave parameters efficiently.



### 5.4.1 Getting initial signal parameter estimators

Although the result of this LS fitting is used only to start the ML optimization, the parameters of this fitting also must be set. This dialog box is very similar to the one shown in Section 5.1. Both the input GUI elements and their function are the same. Note that the samples that are thrown because of the amplitude or time limits will not be considered either in the LS or in the ML fit. The initial estimators appear in the dialog box of ML estimation before the iteration is started (Figure 11).

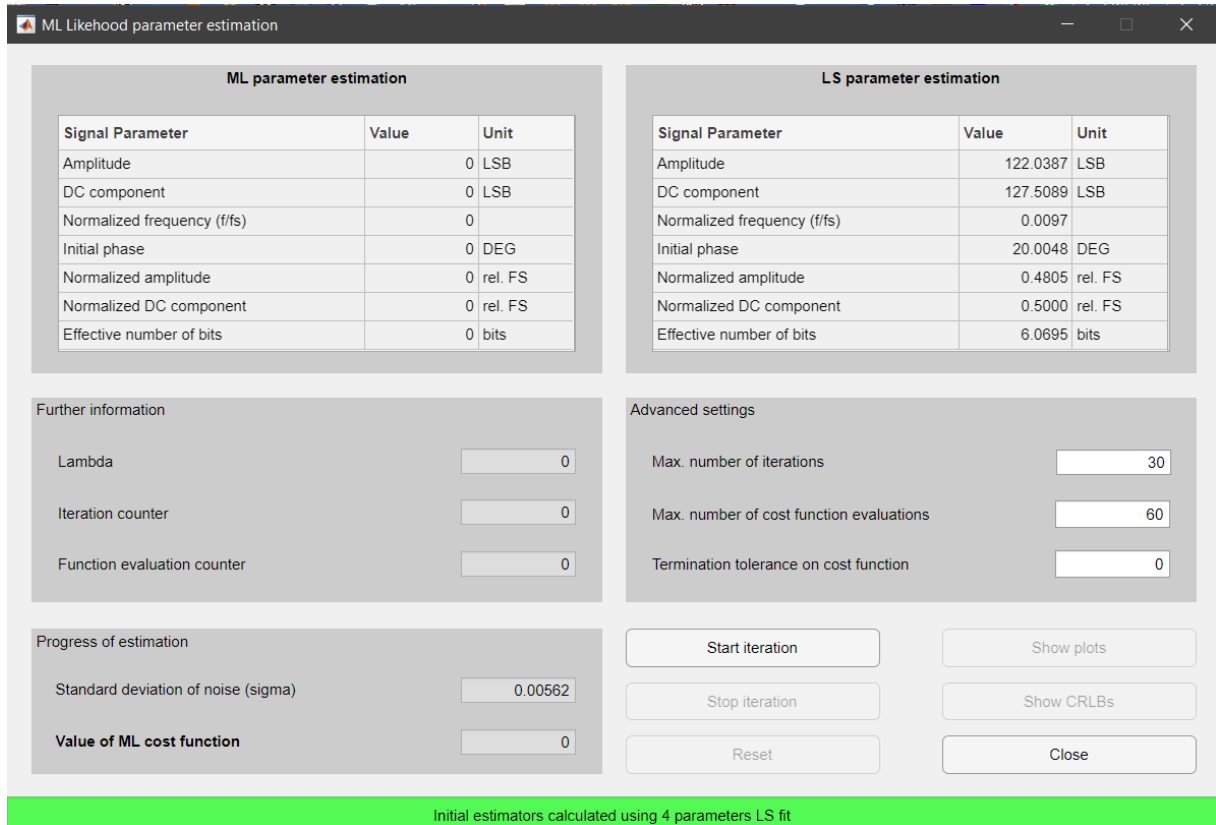


Figure 11: ML estimation dialog box before the iteration is started

### 5.4.2 Progress of optimization: iteration cycles

The iteration can be started in the ML estimation dialog box using the "Start" pushbutton. The changed dialog box is shown in Figure 12.

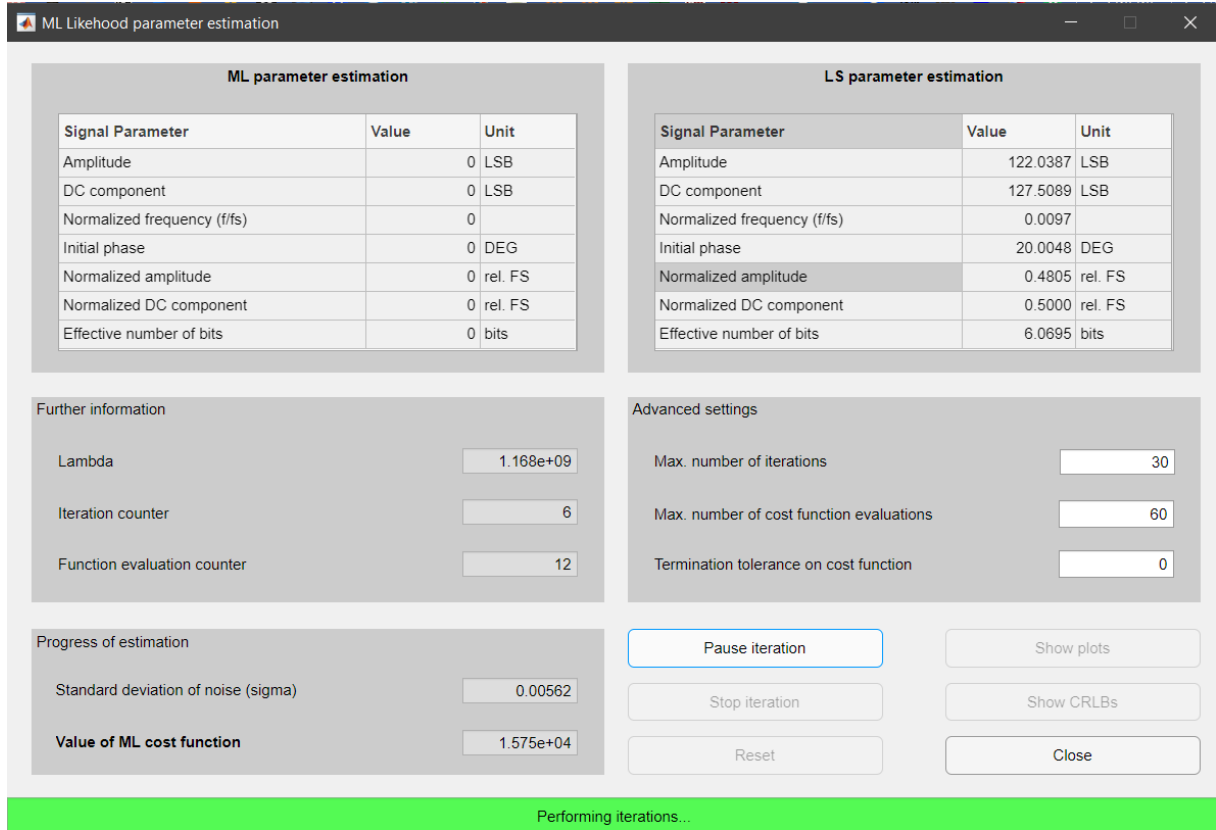


Figure 12: ML estimation dialog box during the iteration

During the computation, the label and the function of this pushbutton change to "Pause". The "Stop iteration" button is used to stop the entire optimization process. The "Reset" button resets the signal parameters to the initial values, and parameters of optimization can be set to different values. The advanced settings are the following:

- **Maximum number of iterations:** used to limit length of optimization.
- **Maximum number of cost function evaluations:** the length of an iteration cycle is not deterministic: depending on the behavior of the cost function, there may be one or more function and derivative evaluations in an iteration cycle. Limiting the number of function evaluations provides a more precise time boundary.
- **Termination tolerance on cost function:** optimization terminates when the difference between the cost function in two consecutive iteration cycles is less than the tolerance specified here.

In the dialog box, the process of optimization can be traced. The actual values of parameter estimators and the value of the ML cost function are displayed in each iteration cycle. The constant that determines the size and direction of the Levenberg-Marquardt step ("lambda"), and the number of iterations and cost function evaluations also appear.

### 5.4.3 Results of optimization: comparison of ML and LS estimators

After the optimization is performed both the LS and ML signal parameter estimators are available. The results can be compared, as demonstrated in Figure 13.

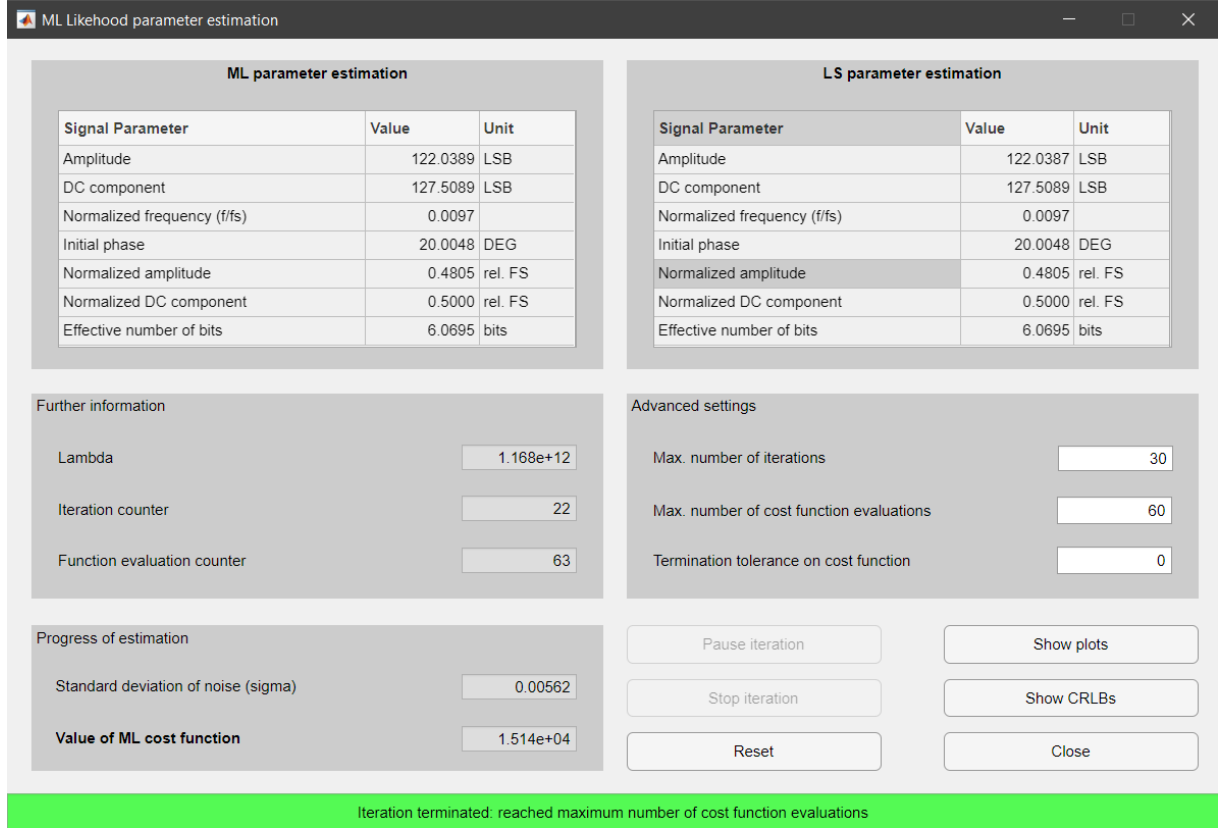


Figure 13: Comparison of ML and LS estimators

In this dialog box the estimated signal and ADC parameters appear. The amplitude and the DC component are displayed in ADC code units (LSB), and also in the percentage of the full scale (FS). The initial phase is displayed in degrees. The frequency of the sine wave is displayed relative to the sampling frequency. The effective number of bits (ENOB) is calculated using the difference between the record and the fitted sine wave (the residuals). **Note that the ENOB calculated using the ML estimators cannot be higher than the ENOB calculated using the LS estimators, because the LS optimization minimizes the root mean square of the residuals, thus maximizing the ENOB value.**

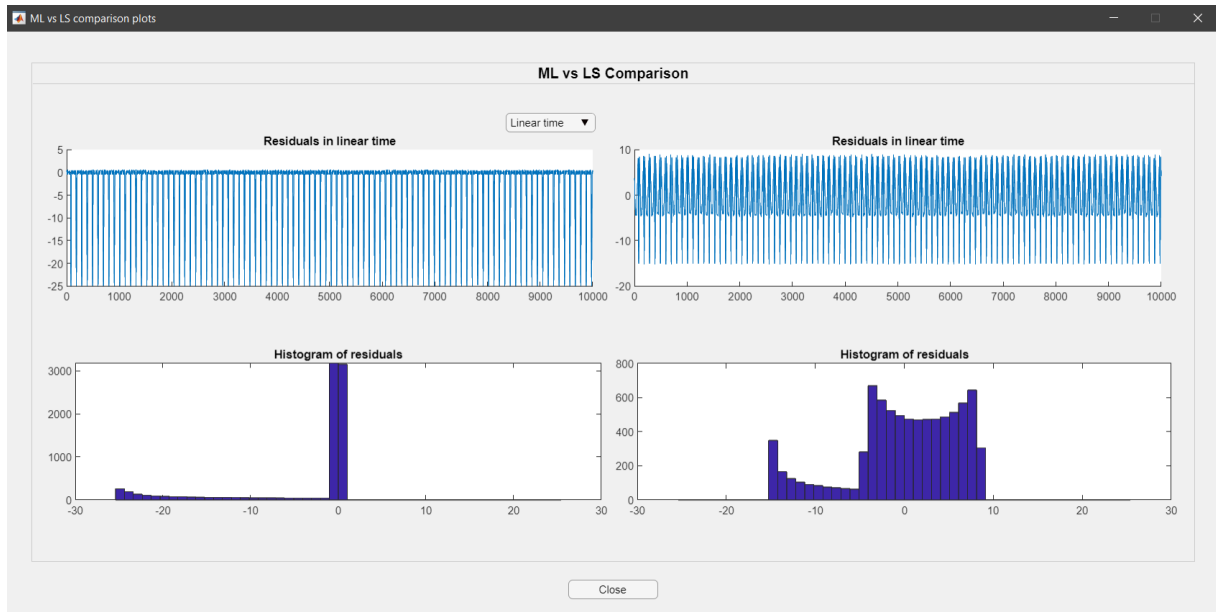


Figure 14: ML and LS comparison residuals

The residuals (Figure 14) can be opened with the “Show plots” pushbutton. There are three ways to plot the residuals:

- **Residuals in linear time:** the residuals are displayed according to their position in the record.
- **Mod T plot of residuals:** the residuals are displayed according to their position in the phase space: phase position of each residual is calculated using the estimated frequency of the signal, and residuals are sorted with respect to this phase value. A reference sine wave is also displayed: the initial phase of this sine wave is the estimated initial phase of the excitation signal.
- **Histogram of residuals:** histogram is calculated to observe statistical moments of residuals.

The linear time and the mod T plot of residuals do not appear in the same time: the content of the higher graph can be switched using a popup menu on the side of ML estimation. This switch changes the appearance of both sides of the window.

Using the "Show CRLB" pushbutton the most important calculated values of the Cramer-Rao Lower Bound regarding the estimated signal parameters appear in a dialog box (see Figure 15). The CRLB values are scaled to units of LSB, and relative frequency ( $f/f_s$ )

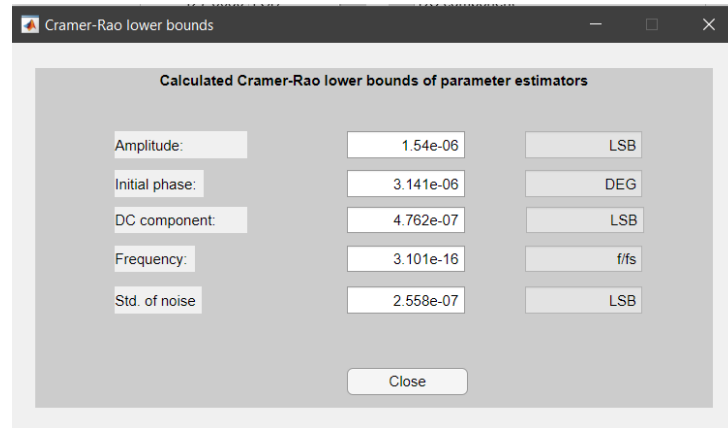


Figure 15: Most important values of the calculated CRLBs

## 5.5 Approximate maximum likelihood (AML) estimation of signal and quantizer parameters

This method parameterizes the integral nonlinearity of the quantizer under test and optimizes the ML cost function with respect to the parameters of the excitation signal and the parameters of the quantizer nonlinearity, as well. In this version of the ADCTest toolbox, the integral nonlinearity of the quantizer is approximated using Fourier coefficients (15 real parameters). To perform AML estimation the user has to press the "Perform AML fit" pushbutton (see Figure 16). The potential warnings or errors regarding AML fit can be read using the "Show warnings" and "Show errors" pushbuttons. Note that if neither warnings nor errors appear, the corresponding pushbuttons are disabled.

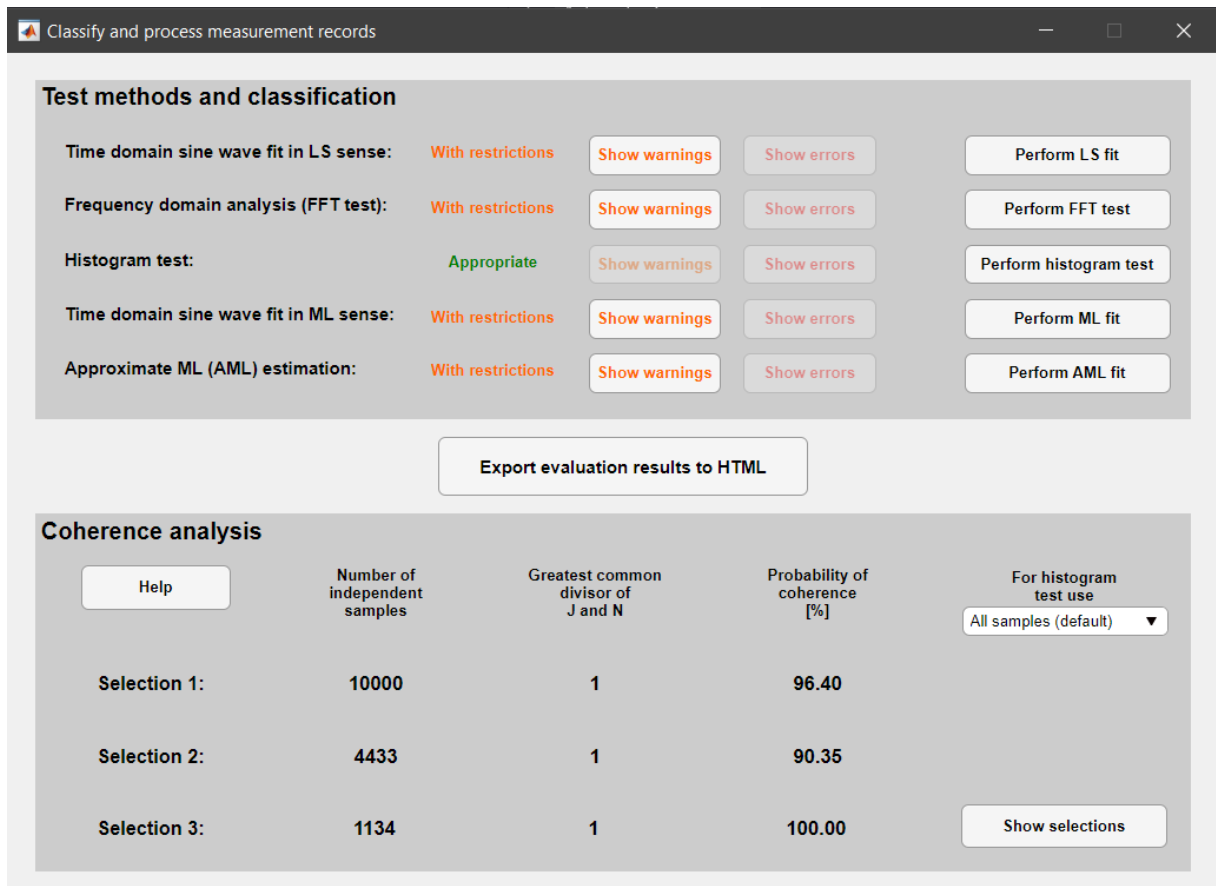


Figure 16: How to choose AML fit

Choosing the AML option calls the standard four parameter least squares fit to achieve initial estimators for the excitation signal parameters. After setting the options and parameters of the LS fit (default values usually provide good performance) we can press the pushbutton entitled "Get initial estimators for approximate ML (AML) fit". This action starts the optimization process which can be followed (and interrupted if necessary) by the user via the GUI (see Figure 17).

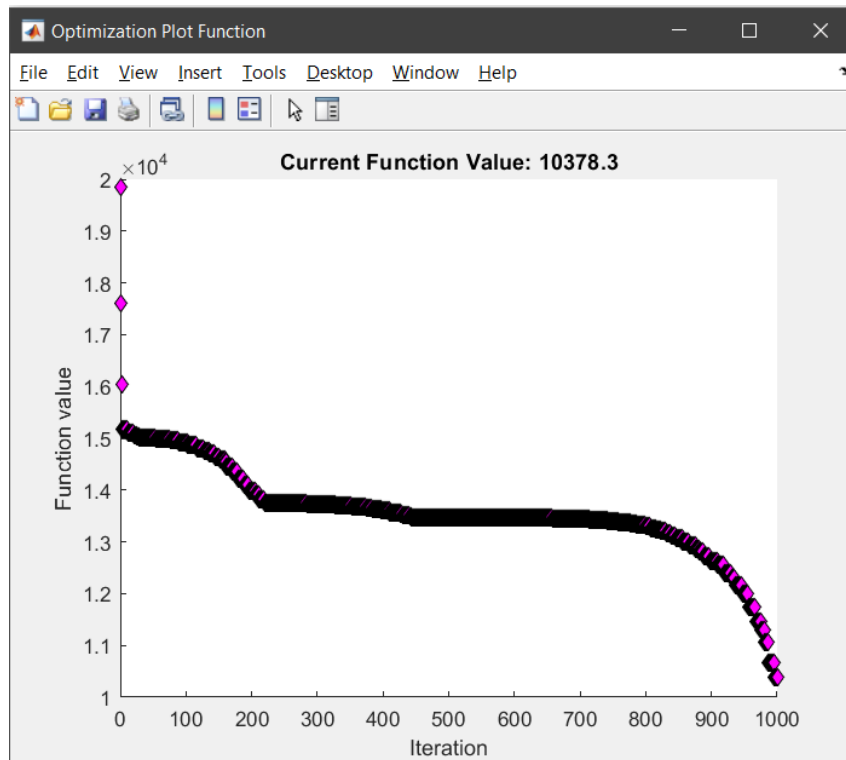


Figure 17: Optimization of the cost function for AML estimation

The results of the AML estimation are displayed in a dialog box (see Figure 18). This dialog box itemizes the main parameters of the optimization process (number of iterations, number of cost function evaluations, duration of optimization, reason of termination, initial and final values of the cost function). The most important quality measures (the effective number of bits and the maximum integral nonlinearity) of the ADC under test are displayed as well. The estimated integral nonlinearity and the "Mod T" plot of the residuals (residuals of the sine wave fitting in the time domain) also appear in this dialog box.

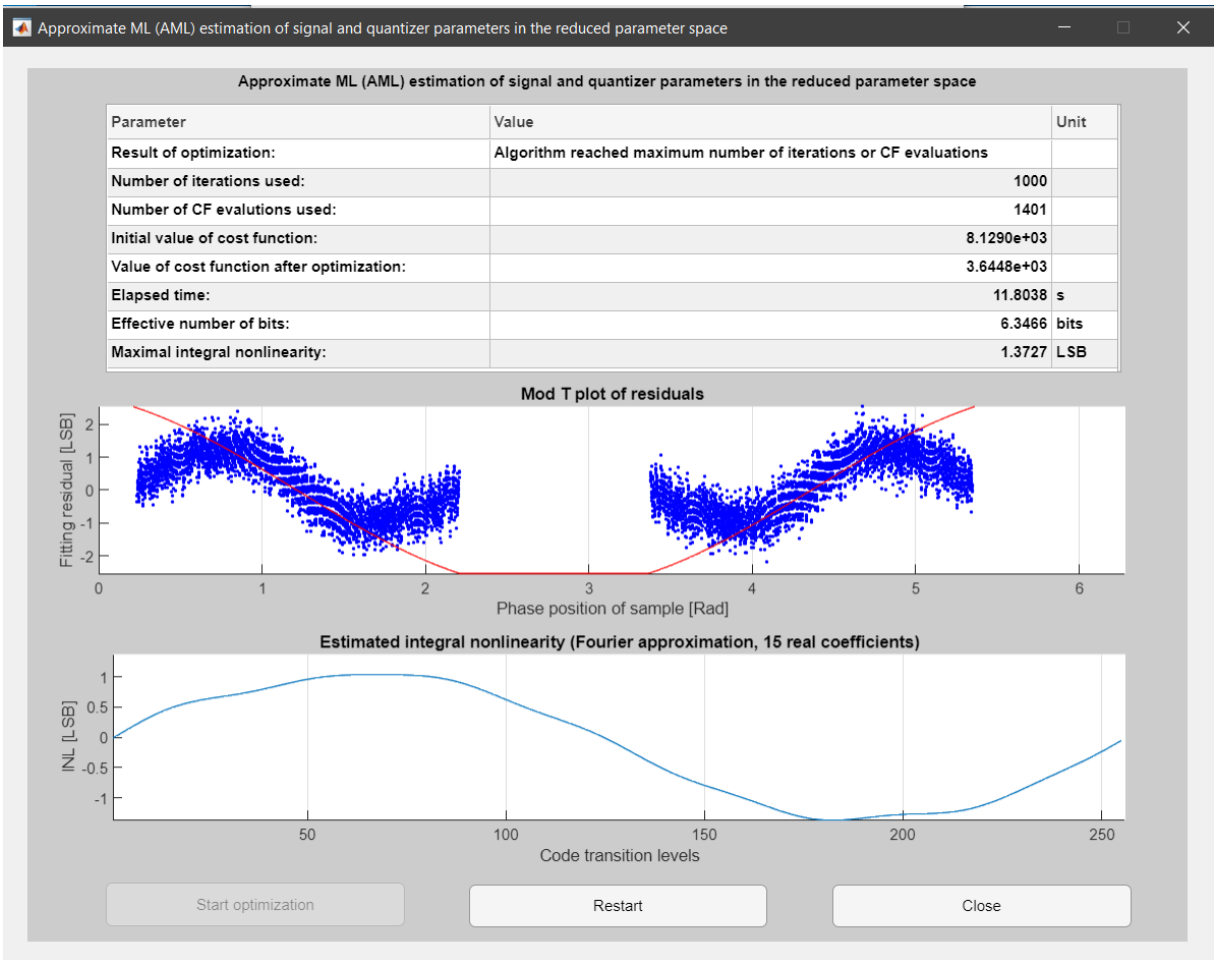


Figure 18: Results of AML estimation



## 6 Calculation of the likelihood (log-likelihood) function and the first and second order derivatives

### 6.1 The likelihood and log-likelihood function

For maximum likelihood estimation of ADC testing, the following model has been developed. The converter is described with a set of code transition levels. Transition level  $T[k]$  is the value of the input voltage, which results in code  $k - 1$  with a probability of 50%, and code  $k$  with a probability of 50% as well. The  $N$ -bit quantizer provides codes from 0 up to  $2^N - 1$ , and has  $2^N - 1$  transition levels. The reduced full scale of the converter is the voltage range between  $T[1]$  and  $T[2^N - 1]$ . Voltage values above the highest transition level result in code  $2^N - 1$  and voltages below the lowest transition level result in code 0. Quantization can be described with a function  $q(x)$  where

$$q(x) = \begin{cases} 0, & \text{if } x < T[1] \\ m, & \text{if } T[m] < x < T[m + 1] \\ 2^N - 1, & \text{if } x > T[2^N - 1] \end{cases} \quad (1)$$

The sinusoidal excitation signal can be described with four parameters:

$$x(t) = A \cos(2\pi ft) + B \sin(2\pi ft) + C \quad (2)$$

where  $A$  is the cosine coefficient,  $B$  is the sine coefficient, and  $C$  denotes the DC component of the signal. The frequency of the sine wave is denoted with  $f$ . The electronic noise of the devices and the imperfections of the measurement environment are modeled with additional noise on the excitation signal. The most manifest is to assume Gaussian noise with zero mean and  $\sigma$  standard deviation. Let  $n(t)$  denote the realization of the additive noise. In this model the spectrum of the noise is white, so  $n(\tau_1)$  and  $n(\tau_2)$  are independent if  $\tau_1 \neq \tau_2$ . The most manifest is to assume Gaussian noise with zero mean and  $\sigma$  standard deviation. Let  $n(t)$  denote the realization of the additive noise. In this model the spectrum of the noise is white, so  $n(\tau_1)$  and  $n(\tau_2)$  are independent, if  $\tau_1 \neq \tau_2$ .

This noisy sine wave is quantized and sampled (the sequence is interchangeable). Thus, the output of the ADC can be described this way:

$$y(k) = q(x(t_k) + n(t_k)) \quad (3)$$

where  $t_k$  denotes the  $k_{th}$  sampling time instant ( $k = 1..M$ ).

The parameters of the model to be estimated are the followings:

- The code transition levels of the quantizer:  $T[1], T[2], \dots, T[2^N - 1]$
- The cosine coefficient of the sine wave:  $A$
- The sine coefficient of the sine wave:  $B$
- The DC component of the sine wave:  $C$
- The frequency of the sine wave:  $f$
- The standard deviation of noise on the excitation signal:  $\sigma$

As uniform sampling is assumed (effects of incidental nonideal sampling are not considered in this model), the frequency of the sine wave can be described using the angular frequency normalized to the sampling frequency:

$$\theta = \omega T_s = 2\pi \frac{f}{f_s} \quad (4)$$

where  $T_s$  is the sampling time, and  $f_s$  denotes the sampling frequency. Thus the parameter vector to be estimated is the following:

$$\mathbf{p}^T = [A \ B \ C \ \theta \ \sigma \ T[1] \ T[2] \ \dots \ T[2^N - 2] \ T[2^N - 1]] \quad (5)$$

To express the likelihood of the parameters, it is necessary to introduce a vector of discrete random variables, denoted by  $\mathbf{Y}$ . Value  $Y(k)$  belongs to the  $k^{\text{th}}$  sample of the measurement record and can assume  $2^N$  values: it can be any of the output codes of the ADC from 0 to  $2^N - 1$  with a given probability. These probabilities can be described using the error function:

$$\text{erf}(x) = \frac{2}{\pi} \int_0^x e^{-z^2} dz \quad (6)$$

$$P(Y(k) = 0) = \frac{1}{2} \left[ \text{erf} \left( \frac{T[1] - x(t_k)}{\sigma\sqrt{2}} \right) + 1 \right] \quad (7)$$

$$P(Y(k) = 2^N - 1) = \frac{1}{2} \left[ 1 - \text{erf} \left( \frac{T[2^N - 1] - x(t_k)}{\sigma\sqrt{2}} \right) \right] \quad (8)$$

$$P(Y(k) = l) = \frac{1}{2} \left[ \text{erf} \left( \frac{T[l + 1] - x(t_k)}{\sigma\sqrt{2}} \right) - \text{erf} \left( \frac{T[l] - x(t_k)}{\sigma\sqrt{2}} \right) \right] \quad (9)$$

where  $l = 1..2^N - 2$ .

To avoid using three different cases it is useful to define two "virtual" transition levels of the ADC:  $T[0] = -\infty$  and  $T[2^N] = +\infty$ . This way, the value of  $Y(k)$  can be expressed in one equation:

$$P(Y(k) = l) = \frac{1}{2} \left[ \text{erf} \left( \frac{T[l + 1] - x(t_k)}{\sigma\sqrt{2}} \right) - \text{erf} \left( \frac{T[l] - x(t_k)}{\sigma\sqrt{2}} \right) \right] \quad (10)$$

where  $l = 0..2^N - 1$ .

The likelihood function for the entire measurement is:

$$L(\mathbf{p}) = \prod_{k=1}^M P(Y(k) = y(k)) \quad (11)$$

where  $y(k)$  is the  $k^{\text{th}}$  sample of the digital record. Merging the equations above, one can express the likelihood function this way:

$$L(\mathbf{p}) = \prod_{k=1}^M \frac{1}{2} \left[ \text{erf} \left( \frac{T[y(k) + 1] - x(t_k)}{\sigma\sqrt{2}} \right) - \text{erf} \left( \frac{T[y(k)] - x(t_k)}{\sigma\sqrt{2}} \right) \right] \quad (12)$$

For computations it is feasible to define a cost function, which is the negative log-likelihood function:

$$\text{CF}(\mathbf{p}) = -\ln L(\mathbf{p}) = M \cdot \ln(2) - \sum_{k=1}^M \ln \left[ \text{erf} \left( \frac{T[y(k)+1] - x(t_k)}{\sigma\sqrt{(2)}} \right) - \text{erf} \left( \frac{T[y(k)] - x(t_k)}{\sigma\sqrt{(2)}} \right) \right] \quad (13)$$

## 6.2 First order derivatives

To ease the expression of the formulas, let us use the following notation. Let  $\arg(k)$  denote the argument of the natural logarithm function in the  $k^{\text{th}}$  element of the cost function. This means that  $\arg(k)$  is two times larger than the probability of measuring  $y(k)$  for the  $k^{\text{th}}$  sample, assuming parameters  $\mathbf{p}$ .

$$\arg(k) = \text{erf} \left( \frac{T[y(k)+1] - x(k)}{\sqrt{2}\sigma} \right) - \text{erf} \left( \frac{T[y(k)] - x(k)}{\sqrt{2}\sigma} \right) \quad (14)$$

The first order partial derivatives of the cost function are expressed below:

$$\frac{\partial \text{CF}}{\partial A} = - \sum_{k=1}^M \frac{1}{\arg(k)} \frac{2}{\sqrt{\pi}} \left( e^{-\left(\frac{T[y(k)]-x(k)}{\sqrt{2}\sigma}\right)^2} \cdot \frac{\cos(k\theta)}{\sqrt{2}\sigma} - e^{-\left(\frac{T[y(k)+1]-x(k)}{\sqrt{2}\sigma}\right)^2} \cdot \frac{\cos(k\theta)}{\sqrt{2}\sigma} \right) \quad (15)$$

$$\frac{\partial \text{CF}}{\partial B} = - \sum_{k=1}^M \frac{1}{\arg(k)} \frac{2}{\sqrt{\pi}} \left( e^{-\left(\frac{T[y(k)]-x(k)}{\sqrt{2}\sigma}\right)^2} \cdot \frac{\sin(k\theta)}{\sqrt{2}\sigma} - e^{-\left(\frac{T[y(k)+1]-x(k)}{\sqrt{2}\sigma}\right)^2} \cdot \frac{\sin(k\theta)}{\sqrt{2}\sigma} \right) \quad (16)$$

$$\frac{\partial \text{CF}}{\partial C} = - \sum_{k=1}^M \frac{1}{\arg(k)} \frac{2}{\sqrt{\pi}} \left( e^{-\left(\frac{T[y(k)]-x(k)}{\sqrt{2}\sigma}\right)^2} \cdot \frac{1}{\sqrt{2}\sigma} - e^{-\left(\frac{T[y(k)+1]-x(k)}{\sqrt{2}\sigma}\right)^2} \cdot \frac{1}{\sqrt{2}\sigma} \right) \quad (17)$$

$$\frac{\partial \text{CF}}{\partial \theta} = - \sum_{k=1}^M \frac{1}{\arg(k)} \cdot \frac{\partial \arg(k)}{\partial \theta} \quad (18)$$

where

$$\frac{\partial \arg(k)}{\partial \theta} = \frac{2}{\sqrt{\pi}} \left( e^{-\left(\frac{T[y(k)+1]-x(k)}{\sqrt{2}\sigma}\right)^2} \cdot \frac{A \sin(k\theta)k - B \cos(k\theta)k}{\sqrt{2}\sigma} - e^{-\left(\frac{T[y(k)]-x(k)}{\sqrt{2}\sigma}\right)^2} \cdot \frac{A \sin(k\theta)k - B \cos(k\theta)k}{\sqrt{2}\sigma} \right) \quad (19)$$

$$\frac{\partial \text{CF}}{\partial \sigma} = - \sum_{k=1}^M \frac{1}{\arg(k)} \cdot \frac{\partial \arg(k)}{\partial \sigma} \quad (20)$$

where

$$\frac{\partial \arg(k)}{\partial \sigma} = \frac{2}{\sqrt{\pi}} \left( e^{-\left(\frac{T[y(k)]-x(k)}{\sqrt{2}\sigma}\right)^2} \cdot \frac{T[y(k)] - x(k)}{\sqrt{2}\sigma^2} - e^{-\left(\frac{T[y(k)+1]-x(k)}{\sqrt{2}\sigma}\right)^2} \cdot \frac{T[y(k)+1] - x(k)}{\sqrt{2}\sigma^2} \right) \quad (21)$$

The partial derivatives with respect to transition levels are the followings:

$$\frac{\partial \text{CF}}{\partial T[l]} = - \sum_{k=1}^M \frac{1}{\arg(k)} \cdot \frac{\partial \arg(k)}{\partial T[l]} \quad (22)$$

where

$$\frac{\partial \arg(k)}{\partial T[l]} = \begin{cases} \frac{2}{\sqrt{\sqrt{\pi}}} \cdot e^{-\left(\frac{T[l]-x(k)}{\sqrt{2}\sigma}\right)^2} \cdot \frac{T[l]-x(k)}{\sigma^2}, & \text{if } y(k) = 1 - 1 \\ -\frac{2}{\sqrt{\sqrt{\pi}}} \cdot e^{-\left(\frac{T[l]-x(k)}{\sqrt{2}\sigma}\right)^2} \cdot \frac{T[l]-x(k)}{\sigma^2}, & \text{if } y(k) = 1 \\ 0, & \text{otherwise} \end{cases} \quad (23)$$

### 6.3 Second order derivatives

The general form of the elements of the Fisher information matrix is the following:

$$I(\mathbf{p})_{i,j} = \sum_{k=1}^M \frac{1}{\arg^2(k)} \cdot \frac{\partial \arg(k)}{\partial p_i} \cdot \frac{\partial \arg(k)}{\partial p_j} - \sum_{k=1}^M \frac{1}{\arg(k)} \cdot \frac{\partial^2 \arg(k)}{\partial p_i \partial p_j} \quad (24)$$

To ease notation let us use the following terms

$$Q_u(k) = \frac{T[y(k) + 1] - A \cos(k\theta) - B \sin(k\theta) - C}{\sqrt{2}\sigma} \quad (25)$$

$$Q_l(k) = \frac{T[y(k)] - A \cos(k\theta) - B \sin(k\theta) - C}{\sqrt{2}\sigma} \quad (26)$$

Since the Fisher information matrix is symmetric, only the diagonal elements and the elements over the diagonal are itemized. The derivatives including the  $l^{\text{th}}$  code transition level ( $T[l]$ ) appear in the  $l + 5^{\text{th}}$  rows and columns respectively.

### 6.4 I(1,1)

$$\frac{\partial^2 \text{CF}}{\partial A^2} = \sum_{k=1}^M \frac{1}{\arg^2(k)} \cdot \left( \frac{\partial \arg(k)}{\partial A} \right)^2 - \sum_{k=1}^M \frac{1}{\arg(k)} \cdot \frac{\partial^2 \arg(k)}{\partial A^2} \quad (27)$$

where

$$\frac{\partial^2 \arg(k)}{\partial A^2} = \frac{-4}{\sqrt{\pi}} e^{-Q_u^2(k)} \cdot Q_u(k) \cdot \frac{\cos^2(k\theta)}{2\sigma^2} - \frac{-4}{\sqrt{\pi}} e^{-Q_l^2(k)} \cdot Q_l(k) \cdot \frac{\cos^2(k\theta)}{2\sigma^2} \quad (28)$$

### 6.5 I(1,2)

$$\frac{\partial^2 \text{CF}}{\partial A \partial B} = \sum_{k=1}^M \frac{1}{\arg^2(k)} \cdot \frac{\partial \arg(k)}{\partial B} \cdot \frac{\partial \arg(k)}{\partial A} - \sum_{k=1}^M \frac{1}{\arg(k)} \cdot \frac{\partial^2 \arg(k)}{\partial A \partial B} \quad (29)$$

where

$$\frac{\partial^2 \arg(k)}{\partial A \partial B} = \frac{-4}{\sqrt{\pi}} e^{-Q_u^2(k)} \cdot Q_u(k) \cdot \frac{\cos(k\theta) \cdot \sin(k\theta)}{2\sigma^2} - \frac{-4}{\sqrt{\pi}} e^{-Q_l^2(k)} \cdot Q_l(k) \cdot \frac{\cos(k\theta) \cdot \sin(k\theta)}{2\sigma^2} \quad (30)$$

### 6.6 I(1,3)

$$\frac{\partial^2 \text{CF}}{\partial A \partial C} = \sum_{k=1}^M \frac{1}{\arg^2(k)} \cdot \frac{\partial \arg(k)}{\partial C} \cdot \frac{\partial \arg(k)}{\partial A} - \sum_{k=1}^M \frac{1}{\arg(k)} \cdot \frac{\partial^2 \arg(k)}{\partial A \partial C} \quad (31)$$

where

$$\frac{\partial^2 \arg(k)}{\partial A \partial C} = \frac{-4}{\sqrt{\pi}} e^{-Q_u^2(k)} \cdot Q_u(k) \cdot \frac{\cos(k\theta)}{2\sigma^2} - \frac{-4}{\sqrt{\pi}} e^{-Q_l^2(k)} \cdot Q_l(k) \cdot \frac{\cos(k\theta)}{2\sigma^2} \quad (32)$$

### 6.7 I(1,4)

$$\frac{\partial^2 \text{CF}}{\partial A \partial \theta} = \sum_{k=1}^M \frac{1}{\arg^2(k)} \cdot \frac{\partial \arg(k)}{\partial \theta} \cdot \frac{\partial \arg(k)}{\partial A} - \sum_{k=1}^M \frac{1}{\arg(k)} \cdot \frac{\partial^2 \arg(k)}{\partial A \partial \theta} \quad (33)$$

where

$$\frac{\partial^2 \arg(k)}{\partial A \partial \theta} = \frac{4}{\sqrt{\pi}} e^{-Q_u^2(k)} \cdot Q_u(k) \cdot \frac{\cos(k\theta)(A \sin(k\theta) \cdot k - B \cos(k\theta) \cdot k)}{2\sigma^2} + \frac{2}{\sqrt{\pi}} \cdot e^{-Q_u^2(k)} \frac{\sin(k\theta) \cdot k}{\sqrt{2}\sigma} - \quad (34)$$

$$- \frac{4}{\sqrt{\pi}} e^{-Q_l^2(k)} \cdot Q_l(k) \cdot \frac{\cos(k\theta)(A \sin(k\theta) \cdot k - B \cos(k\theta) \cdot k)}{2\sigma^2} + \frac{2}{\sqrt{\pi}} \cdot e^{-Q_l^2(k)} \frac{\sin(k\theta) \cdot k}{\sqrt{2}\sigma} \quad (34)$$

### 6.8 I(1,5)

$$\frac{\partial^2 \text{CF}}{\partial A \partial \sigma} = \sum_{k=1}^M \frac{1}{\arg^2(k)} \cdot \frac{\partial \arg(k)}{\partial \sigma} \cdot \frac{\partial \arg(k)}{\partial A} - \sum_{k=1}^M \frac{1}{\arg(k)} \cdot \frac{\partial^2 \arg(k)}{\partial A \partial \sigma} \quad (35)$$

where

$$\frac{\partial^2 \arg(k)}{\partial A \partial \sigma} = \frac{-4}{\sqrt{\pi}} \cdot e^{-Q_u^2(k)} \cdot \frac{Q_u^2(k) \cos(k\theta)}{\sqrt{2}\sigma^2} + \frac{2}{\sqrt{\pi}} \cdot e^{-Q_u^2(k)} \cdot \frac{\cos(k\theta)}{\sqrt{2}\sigma^2} - \quad (36)$$

$$- \frac{4}{\sqrt{\pi}} \cdot e^{-Q_l^2(k)} \cdot \frac{Q_l^2(k) \cos(k\theta)}{\sqrt{2}\sigma^2} + \frac{2}{\sqrt{\pi}} \cdot e^{-Q_l^2(k)} \cdot \frac{\cos(k\theta)}{\sqrt{2}\sigma^2} - \quad (36)$$

### 6.9 I(2,2)

$$\frac{\partial^2 \text{CF}}{\partial B^2} = \sum_{k=1}^M \frac{1}{\arg^2(k)} \cdot \frac{\partial \arg(k)}{\partial B} \cdot \frac{\partial \arg(k)}{\partial B} - \sum_{k=1}^M \frac{1}{\arg(k)} \cdot \frac{\partial^2 \arg(k)}{\partial B^2} \quad (37)$$

where

$$\frac{\partial^2 \arg(k)}{\partial B^2} = \frac{-4}{\sqrt{\pi}} \cdot e^{-Q_u^2(k)} \cdot \frac{Q_u(k) \cdot \sin^2(k\theta)}{2\sigma^2} + \frac{4}{\sqrt{\pi}} \cdot e^{-Q_l^2(k)} \cdot \frac{Q_l(k) \cdot \sin^2(k\theta)}{2\sigma^2} \quad (38)$$

### 6.10 I(2,3)

$$\frac{\partial^2 \text{CF}}{\partial B \partial C} = \sum_{k=1}^M \frac{1}{\arg^2(k)} \cdot \frac{\partial \arg(k)}{\partial C} \cdot \frac{\partial \arg(k)}{\partial B} - \sum_{k=1}^M \frac{1}{\arg(k)} \cdot \frac{\partial^2 \arg(k)}{\partial B \partial C} \quad (39)$$

where

$$\frac{\partial^2 \arg(k)}{\partial B \partial C} = \frac{-4}{\sqrt{\pi}} \cdot e^{-Q_u^2(k)} \cdot \frac{Q_u(k) \cdot \sin(k\theta)}{2\sigma^2} + \frac{4}{\sqrt{\pi}} \cdot e^{-Q_l^2(k)} \cdot \frac{Q_l(k) \cdot \sin(k\theta)}{2\sigma^2} \quad (40)$$

### 6.11 I(2,4)

$$\frac{\partial^2 \text{CF}}{\partial B \partial \theta} = \sum_{k=1}^M \frac{1}{\arg^2(k)} \cdot \frac{\partial \arg(k)}{\partial \theta} \cdot \frac{\partial \arg(k)}{\partial B} - \sum_{k=1}^M \frac{1}{\arg(k)} \cdot \frac{\partial^2 \arg(k)}{\partial B \partial \theta} \quad (41)$$

where

$$\frac{\partial^2 \arg(k)}{\partial B \partial \theta} = \frac{4}{\sqrt{\pi}} \cdot e^{-Q_u^2(k)} \cdot \frac{Q_u(k) \cdot (A \sin(k\theta)k - B \cos(k\theta)k) \cdot \sin(k\theta)}{2\sigma^2} - \frac{2}{\sqrt{\pi}} \cdot e^{-Q_u^2(k)} \cdot \frac{\cos(k\theta)k}{\sqrt{2}\sigma} \quad (42)$$

$$- \frac{4}{\sqrt{\pi}} \cdot e^{-Q_l^2(k)} \cdot \frac{Q_l(k) \cdot (A \sin(k\theta)k - B \cos(k\theta)k) \cdot \sin(k\theta)}{2\sigma^2} + \frac{2}{\sqrt{\pi}} \cdot e^{-Q_l^2(k)} \cdot \frac{\cos(k\theta)k}{\sqrt{2}\sigma} \quad (42)$$

### 6.12 I(2,5)

$$\frac{\partial^2 \text{CF}}{\partial B \partial \sigma} = \sum_{k=1}^M \frac{1}{\arg^2(k)} \cdot \frac{\partial \arg(k)}{\partial \sigma} \cdot \frac{\partial \arg(k)}{\partial B} - \sum_{k=1}^M \frac{1}{\arg(k)} \cdot \frac{\partial^2 \arg(k)}{\partial B \partial \sigma} \quad (43)$$

where

$$\frac{\partial^2 \arg(k)}{\partial B \partial \sigma} = \frac{-4}{\sqrt{\pi}} \cdot e^{-Q_u^2(k)} \cdot \frac{Q_u^2(k) \cdot \sin(k\theta)}{\sqrt{2}\sigma^2} + \frac{2}{\sqrt{\pi}} \cdot e^{-Q_u^2(k)} \cdot \frac{\sin(k\theta)}{\sqrt{2}\sigma^2} + \quad (44)$$

$$+ \frac{4}{\sqrt{\pi}} \cdot e^{-Q_l^2(k)} \cdot \frac{Q_l^2(k) \cdot \sin(k\theta)}{\sqrt{2}\sigma^2} - \frac{2}{\sqrt{\pi}} \cdot e^{-Q_l^2(k)} \cdot \frac{\sin(k\theta)}{\sqrt{2}\sigma^2} \quad (44)$$

### 6.13 I(3,3)

$$\frac{\partial^2 \text{CF}}{\partial C^2} = \sum_{k=1}^M \frac{1}{\arg^2(k)} \cdot \frac{\partial \arg(k)}{\partial C} \cdot \frac{\partial \arg(k)}{\partial C} - \sum_{k=1}^M \frac{1}{\arg(k)} \cdot \frac{\partial^2 \arg(k)}{\partial C^2} \quad (45)$$

where

$$\frac{\partial^2 \arg(k)}{\partial C^2} = \frac{-4}{\sqrt{\pi}} \cdot e^{-Q_u^2(k)} \cdot \frac{Q_u(k)}{2\sigma^2} + \frac{4}{\sqrt{\pi}} \cdot e^{-Q_u^2(k)} \cdot \frac{Q_l(k)}{2\sigma^2} \quad (46)$$

### 6.14 I(3,4)

$$\frac{\partial^2 \text{CF}}{\partial C \partial \theta} = \sum_{k=1}^M \frac{1}{\arg^2(k)} \cdot \frac{\partial \arg(k)}{\partial \theta} \cdot \frac{\partial \arg(k)}{\partial C} - \sum_{k=1}^M \frac{1}{\arg(k)} \cdot \frac{\partial^2 \arg(k)}{\partial C \partial \theta} \quad (47)$$

where

$$\frac{\partial^2 \arg(k)}{\partial C \partial \theta} = \frac{-4}{\sqrt{\pi}} \cdot e^{-Q_u^2(k)} \cdot \frac{Q_u(k) \cdot (A \sin(k\theta)k - B \cos(k\theta)k)}{2\sigma^2} + \frac{4}{\sqrt{\pi}} \cdot e^{-Q_l^2(k)} \cdot \frac{Q_l(k) \cdot (A \sin(k\theta)k - B \cos(k\theta)k)}{2\sigma^2} \quad (48)$$

### 6.15 I(3,5)

$$\frac{\partial^2 \text{CF}}{\partial C \partial \sigma} = \sum_{k=1}^M \frac{1}{\arg^2(k)} \cdot \frac{\partial \arg(k)}{\partial \sigma} \cdot \frac{\partial \arg(k)}{\partial C} - \sum_{k=1}^M \frac{1}{\arg(k)} \cdot \frac{\partial^2 \arg(k)}{\partial C \partial \sigma} \quad (49)$$

where

$$\frac{\partial^2 \arg(k)}{\partial C \partial \sigma} = \frac{-4}{\sqrt{\pi}} \cdot e^{-Q_u^2(k)} \cdot \frac{Q_u^2(k)}{\sqrt{2}\sigma^2} + \frac{2}{\sqrt{\pi}} \cdot e^{-Q_u^2(k)} \cdot \frac{1}{\sqrt{2}\sigma^2} + \quad (50)$$

$$+ \frac{4}{\sqrt{\pi}} \cdot e^{-Q_l^2(k)} \cdot \frac{Q_l^2(k)}{\sqrt{2}\sigma^2} - \frac{2}{\sqrt{\pi}} \cdot e^{-Q_l^2(k)} \cdot \frac{1}{\sqrt{2}\sigma^2} \quad (50)$$

### 6.16 I(4,4)

$$\frac{\partial^2 \text{CF}}{\partial \theta^2} = \sum_{k=1}^M \frac{1}{\arg^2(k)} \cdot \frac{\partial \arg(k)}{\partial \theta} \cdot \frac{\partial \arg(k)}{\partial \theta} - \sum_{k=1}^M \frac{1}{\arg(k)} \cdot \frac{\partial^2 \arg(k)}{\partial \theta^2} \quad (51)$$

where

$$\frac{\partial^2 \arg(k)}{\partial \theta^2} = \frac{-4}{\sqrt{\pi}} \cdot e^{-Q_u^2(k)} \cdot \frac{Q_u(k) \cdot (A \sin(k\theta)k - B \cos(k\theta)k)^2}{2\sigma^2} + \frac{2}{\sqrt{\pi}} \cdot e^{-Q_u^2(k)} \cdot \frac{A \cos(k\theta)k^2 + B \sin(k\theta)k^2}{\sqrt{2}\sigma} + \quad (52)$$

$$+ \frac{4}{\sqrt{\pi}} \cdot e^{-Q_l^2(k)} \cdot \frac{Q_l(k) \cdot (A \sin(k\theta)k - B \cos(k\theta)k)^2}{2\sigma^2} - \frac{2}{\sqrt{\pi}} \cdot e^{-Q_l^2(k)} \cdot \frac{A \cos(k\theta)k^2 + B \sin(k\theta)k^2}{\sqrt{2}\sigma} \quad (52)$$

### 6.17 I(4,5)

$$\frac{\partial^2 \text{CF}}{\partial \theta \partial \sigma} = \sum_{k=1}^M \frac{1}{\arg^2(k)} \cdot \frac{\partial \arg(k)}{\partial \sigma} \cdot \frac{\partial \arg(k)}{\partial \theta} - \sum_{k=1}^M \frac{1}{\arg(k)} \cdot \frac{\partial^2 \arg(k)}{\partial \theta \partial \sigma} \quad (53)$$

where

$$\frac{\partial^2 \arg(k)}{\partial \theta \partial \sigma} = \frac{4}{\sqrt{\pi}} \cdot e^{-Q_u^2(k)} \cdot \frac{Q_u^2(k) \cdot (A \sin(k\theta)k - B \cos(k\theta)k)}{\sqrt{2}\sigma^2} - \frac{2}{\sqrt{\pi}} \cdot e^{-Q_u^2(k)} \cdot \frac{A \sin(k\theta)k - B \cos(k\theta)k}{\sqrt{2}\sigma^2} \quad (54)$$

$$- \frac{4}{\sqrt{\pi}} \cdot e^{-Q_l^2(k)} \cdot \frac{Q_l^2(k) \cdot (A \sin(k\theta)k - B \cos(k\theta)k)}{\sqrt{2}\sigma^2} + \frac{2}{\sqrt{\pi}} \cdot e^{-Q_l^2(k)} \cdot \frac{A \sin(k\theta)k - B \cos(k\theta)k}{\sqrt{2}\sigma^2} \quad (54)$$

### 6.18 I(5,5)

$$\frac{\partial^2 \text{CF}}{\partial \sigma^2} = \sum_{k=1}^M \frac{1}{\arg^2(k)} \cdot \frac{\partial \arg(k)}{\partial \sigma} \cdot \frac{\partial \arg(k)}{\partial \sigma} - \sum_{k=1}^M \frac{1}{\arg(k)} \cdot \frac{\partial^2 \arg(k)}{\partial \sigma^2} \quad (55)$$

where

$$\frac{\partial^2 \arg(k)}{\partial \sigma^2} = \frac{-4}{\sqrt{\pi}} \cdot e^{-Q_u^2(k)} \cdot \frac{Q_u^3(k)}{\sigma^2} + \frac{2}{\sqrt{\pi}} \cdot e^{-Q_u^2(k)} \cdot \frac{2Q_u(k)}{\sigma^2} + \quad (56)$$

$$+ \frac{4}{\sqrt{\pi}} \cdot e^{-Q_l^2(k)} \cdot \frac{Q_l^3(k)}{\sigma^2} - \frac{2}{\sqrt{\pi}} \cdot e^{-Q_l^2(k)} \cdot \frac{2Q_l(k)}{\sigma^2} \quad (56)$$

### 6.19 I(1,1+5)

$$\frac{\partial^2 \text{CF}}{\partial A \partial T[l]} = \sum_{k=1}^M \frac{1}{\arg^2(k)} \cdot \frac{\partial \arg(k)}{\partial T[l]} \cdot \frac{\partial \arg(k)}{\partial A} - \sum_{k=1}^M \frac{1}{\arg(k)} \cdot \frac{\partial^2 \arg(k)}{\partial A \partial T[l]} \quad (57)$$

where

$$\frac{\partial^2 \arg(k)}{\partial A \partial T[l]} = \begin{cases} \frac{2}{\sqrt{\pi}} \cdot e^{-Q_u^2} \cdot \frac{\cos(k\theta)}{\sigma^2} \cdot (2Q_u^2 - 1) & \text{if } y[k] = l - 1 \\ -\frac{2}{\sqrt{\pi}} \cdot e^{-Q_l^2} \cdot \frac{\cos(k\theta)}{\sigma^2} \cdot (2Q_l^2 - 1) & \text{if } y[k] = l \\ 0 & \text{otherwise} \end{cases} \quad (58)$$

### 6.20 I(2,1+5)

$$\frac{\partial^2 \text{CF}}{\partial B \partial T[l]} = \sum_{k=1}^M \frac{1}{\arg^2(k)} \cdot \frac{\partial \arg(k)}{\partial T[l]} \cdot \frac{\partial \arg(k)}{\partial B} - \sum_{k=1}^M \frac{1}{\arg(k)} \cdot \frac{\partial^2 \arg(k)}{\partial B \partial T[l]} \quad (59)$$

where

$$\frac{\partial^2 \arg(k)}{\partial B \partial T[l]} = \begin{cases} \frac{2}{\sqrt{\pi}} \cdot e^{-Q_u^2} \cdot \frac{\sin(k\theta)}{\sigma^2} \cdot (2Q_u^2 - 1) & \text{if } y[k] = l - 1 \\ -\frac{2}{\sqrt{\pi}} \cdot e^{-Q_l^2} \cdot \frac{\sin(k\theta)}{\sigma^2} \cdot (2Q_l^2 - 1) & \text{if } y[k] = l \\ 0 & \text{otherwise} \end{cases} \quad (60)$$

### 6.21 I(3,1+5)

$$\frac{\partial^2 \text{CF}}{\partial C \partial T[l]} = \sum_{k=1}^M \frac{1}{\arg^2(k)} \cdot \frac{\partial \arg(k)}{\partial T[l]} \cdot \frac{\partial \arg(k)}{\partial C} - \sum_{k=1}^M \frac{1}{\arg(k)} \cdot \frac{\partial^2 \arg(k)}{\partial C \partial T[l]} \quad (61)$$

where

$$\frac{\partial^2 \arg(k)}{\partial C \partial T[l]} = \begin{cases} \frac{2}{\sqrt{\pi}} \cdot e^{-Q_u^2} \cdot \frac{1}{\sigma^2} \cdot (2Q_u^2 - 1) & \text{if } y[k] = l - 1 \\ -\frac{2}{\sqrt{\pi}} \cdot e^{-Q_l^2} \cdot \frac{1}{\sigma^2} \cdot (2Q_l^2 - 1) & \text{if } y[k] = l \\ 0 & \text{otherwise} \end{cases} \quad (62)$$



### 6.22 I(4,1+5)

$$\frac{\partial^2 \text{CF}}{\partial \theta \partial T[l]} = \sum_{k=1}^M \frac{1}{\arg^2(k)} \cdot \frac{\partial \arg(k)}{\partial T[l]} \cdot \frac{\partial \arg(k)}{\partial \theta} - \sum_{k=1}^M \frac{1}{\arg(k)} \cdot \frac{\partial^2 \arg(k)}{\partial \theta \partial T[l]} \quad (63)$$

where

$$\frac{\partial^2 \arg(k)}{\partial \theta \partial T[l]} = \begin{cases} \frac{2}{\sqrt{\pi}} \cdot e^{-Q_u^2} \cdot \frac{A \sin(k\theta)k - B \cos(k\theta)k}{\sigma^2} \cdot (2Q_u^2 - 1) & \text{if } y[k] = l - 1 \\ -\frac{2}{\sqrt{\pi}} \cdot e^{-Q_l^2} \cdot \frac{A \sin(k\theta)k - B \cos(k\theta)k}{\sigma^2} \cdot (2Q_l^2 - 1) & \text{if } y[k] = l \\ 0 & \text{otherwise} \end{cases} \quad (64)$$

### 6.23 I(5,1+5)

$$\frac{\partial^2 \text{CF}}{\partial \sigma \partial T[l]} = \sum_{k=1}^M \frac{1}{\arg^2(k)} \cdot \frac{\partial \arg(k)}{\partial T[l]} \cdot \frac{\partial \arg(k)}{\partial \sigma} - \sum_{k=1}^M \frac{1}{\arg(k)} \cdot \frac{\partial^2 \arg(k)}{\partial \sigma \partial T[l]} \quad (65)$$

where

$$\frac{\partial^2 \arg(k)}{\partial \sigma \partial T[l]} = \begin{cases} \frac{4\sqrt{2}}{\sqrt{\pi}} \cdot e^{-Q_u^2} \cdot \frac{Q_u}{\sigma^2} \cdot (1 - Q_u) & \text{if } y[k] = l - 1 \\ -\frac{4\sqrt{2}}{\sqrt{\pi}} \cdot e^{-Q_l^2} \cdot \frac{Q_l}{\sigma^2} \cdot (1 - Q_l) & \text{if } y[k] = l \\ 0 & \text{otherwise} \end{cases} \quad (66)$$

### 6.24 I(m+5,1+5)

$$\frac{\partial^2 \text{CF}}{\partial T[m] \partial T[l]} = \sum_{k=1}^M \frac{1}{\arg^2(k)} \cdot \frac{\partial \arg(k)}{\partial T[l]} \cdot \frac{\partial \arg(k)}{\partial T[m]} - \sum_{k=1}^M \frac{1}{\arg(k)} \cdot \frac{\partial^2 \arg(k)}{\partial T[m] \partial T[l]} \quad (67)$$

where

$$\frac{\partial^2 \arg(k)}{\partial T[m] \partial T[l]} = \begin{cases} \frac{\partial^2 \arg(k)}{\partial T[l]^2} & \text{if } m = l \\ 0 & \text{otherwise} \end{cases} \quad (68)$$

### 6.25 I(1+5,1+5)

$$\frac{\partial^2 \text{CF}}{\partial T[l]^2} = \sum_{k=1}^M \frac{1}{\arg^2(k)} \cdot \frac{\partial \arg(k)}{\partial T[l]} \cdot \frac{\partial \arg(k)}{\partial T[l]} - \sum_{k=1}^M \frac{1}{\arg(k)} \cdot \frac{\partial^2 \arg(k)}{\partial T[l]^2} \quad (69)$$

where

$$\frac{\partial^2 \arg(k)}{\partial T[l]^2} = \begin{cases} \frac{2}{\sqrt{\pi}} e^{-Q_u^2} \cdot \frac{1}{\sigma^2} \cdot (2Q_u^2 - 1) & \text{if } y[k] = l - 1 \\ -\frac{2}{\sqrt{\pi}} e^{-Q_l^2} \cdot \frac{1}{\sigma^2} \cdot (2Q_l^2 - 1) & \text{if } y[k] = l \\ 0 & \text{otherwise} \end{cases} \quad (70)$$

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