Leakage Reduction in Frequency-Response Function Measurements

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***Abstract*—This paper analyzes how to reduce leakage errors in frequency-response function (FRF) measurements. First, the nature of leakage errors is revealed; next, windowing methods are analyzed, and a new default window is proposed. Finally, a superior Taylor-series-based method is proposed.**

***Index Terms*—Frequency-response function (FRF) methods, leakage, windows.**

I. Introduction

FREQUENCY-RESPONSE FUNCTION (FRF) measurements of transfer functions are a basic tool in many engineering fields. For random excitations, these measurements are disturbed by leakage and disturbing noise errors. For these reasons, we strongly advise applying periodic excitation signals whenever it is possible [8]. However, in many applications, for psychological or technological reasons, the users prefer to apply random noise excitations. It is well known that for random noise excitations, the FRF measurements are disturbed by leakage (windowing) errors that are induced by the finite length of the measurement window. This paper uses new insights in the nature of these errors to propose improved FRF-measurement techniques.

The classical approach to reduce the leakage errors is based on the use of windows. In the literature, a large number of windows is defined and their properties are intensively studied, keeping essentially spectral analysis applications in mind [2], [6]. In this paper, these properties are analyzed again keeping FRF measurements in mind which leads to new insights, and eventually to the definition of a new window. This allows a reduction of the "leakage errors" on the FRF measurements, while the noise sensitivity is not increased. In the next step, an alternative Taylor-based method is proposed. In its simplest version it reduces to the Hanning window, but with more advanced settings a superior method is found.

II. Hidden Nature of Leakage Errors

Let us consider a stable, causal, discrete- or continuous-time, single-input–single-output linear, time-invariant system with impulse response $g_0(t)$ that is excited with a random input $u_0(t)$

$$y(t) = g_0(t)^* u_0(t) + v(t) = y_0(t) + v(t) \tag{1}$$

with $*$ the convolution, $u_0(t), y_0(t)$ the exact input and output signal, and $v(t)$ disturbing noise. N samples of the input and output are measured at $t = kT_s = k/f_s$. For notational simplicity, we note these measurements as

$$u_0(k) \quad y(k), \qquad \text{with} \quad k = 0, \ldots, N-1. \tag{2}$$

Those results of measurements are used for estimation of the FRF $G_0(l)$ at frequency $f_l = lf_s/N, l = 0, \ldots, N/2$. The discrete Fourier transform (DFT) $U_0(l), Y(l)$ of the input–output signal [3] is

$$X(l) = \frac{1}{N} \sum_{k=0}^{N-1} x(k) e^{-j\frac{2\pi}{N}kl}. \tag{3}$$

The following remarkably simple relation holds between the DFT spectra for $y_0(t) = g_0(t)^* u_0(t)$ [8], [7]

$$Y_0(l) = G_0(l) U_0(l) + T_0(l) \tag{4}$$

with G_0 and T_0 smooth rational functions of the frequency. T_0 can be interpreted as a generalized "transient" term. Some of these ideas were already reported in [5]. Due to the DFT definition (3), $U_0(l), Y_0(l)$, and $Y(l)$ are of order $O(N^{-1/2})$, and the transient $T_0(l)$ is of order $O(N^{-1})$ [8].

In the absence of disturbing noise $v(t) = 0$, the FRF estimate is given by

$$\hat{G}(l) = \frac{Y_0(l)}{U_0(l)} = G_0(l) + \frac{T_0(l)}{U_0(l)} = G_0(l) + O(N^{-1/2}). \tag{5}$$

It is the last term in (5) that causes the leakage in the FRF measurements. The ratio $T_0(l)/U_0(l)$ has a random behavior and looks like noise in FRF measurements because $U_0(l)$ is random. However, this hides the highly structured nature that is described by the smooth function T_0. Windowing methods exploit this smoothness to reduce the leakage errors.

It is common practice to average $\hat{G}(l)$ over multiple measurements [1]

$$\hat{G}^M(l) = \frac{\sum_{m=1}^{M} Y^{[m]}(l) \bar{U}_0^{[m]}(l)}{\sum_{m=1}^{M} U_0^{[m]}(l) \bar{U}_0^{[m]}(l)} \tag{6}$$

where $X^{[m]}(l)$ is the spectrum of the signal in the mth realization of the experiment. This estimate converges for $M \to \infty$ to the solution corresponding to $v(t) = 0$ if the output noise $v(t)$ is not correlated with the input $u_0(t)$

$$\lim_{M \to \infty} \hat{G}^M(l) = \frac{\lim_{M\to\infty} \sum_{m=1}^{M} Y^{[m]}(l) \bar{U}_0^{[m]}(l)}{\lim_{M\to\infty} \sum_{m=1}^{M} U_0^{[m]}(l) \bar{U}_0^{[m]}(l)}. \tag{7}$$

Due to the leakage effects, this limit is still biased as it will be shown later.

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III. Window-Based Methods

A. Hanning Window and Its Spectral Interpretation

The Fourier transform of a discrete-time signal $x(t)$ is an infinite sum $\sum_{t=-\infty}^{\infty} x(t)e^{-j\omega t}$. This infinite sum is restricted to a finite one in the DFT by considering only a finite number of samples. It is calculated on the "windowed" signal

$$x_w(t) = w(t)x(t) \tag{8}$$

with $w(t) = 0$ if t is outside the interval $[0, N-1]$. A large number of different windows is proposed in the literature [6]; here, we focus on the rectangular and the Hanning windows as follows:

1) rectangular (Dirichlet) window (within a proper scaling factor)

$$w(t) = 1, \qquad \text{for} \quad t = 0, 1, \ldots, N-1; \tag{9}$$

2) Hanning window (within a proper scaling factor)

$$w(t) = 0.5 - 0.5\cos(2\pi t/N). \tag{10}$$

There exists a simple relation between the DFT spectra obtained with the Hanning window (X_{Hann}) and the rectangular window (X_{Rect})

$$X_{\text{Hann}}(l) = \frac{2X_{\text{Rect}}(l) - X_{\text{Rect}}(l-1) - X_{\text{Rect}}(l+1)}{4} \tag{11}$$

which is proportional to the second-order difference of the spectrum X_{Rect}.

B. Analysis of the Leakage Errors on the FRF Measurement

1) Rectangular Window: For a rectangular window [1], [3], it is found immediately that in the noiseless case

$$\begin{aligned}\hat{G}_{\text{Rect}}(l) &= \frac{G_0(l)U_0(l)}{U_0(l)} + \frac{T_0(l)}{U_0(l)} \\ &= G_0(l) + O(N^{-1/2}).\end{aligned} \tag{12}$$

The averaged estimate is

$$\hat{G}^m_{\text{Rect}}(l) = G_0(l) + \frac{\frac{1}{M}\sum_{m=1}^{M} \bar{U}_0^{[m]}(l)T_0^{[m]}(l)}{\frac{1}{M}\sum_{m=1}^{M} \bar{U}_0^{[m]}(l)U_0^{[m]}(l)}. \tag{13}$$

a) Systematic errors: The error term in (13) goes not to zero for $M \to \infty$. $T_0^{[m]}$ is the sum of two transient contributions at the beginning and the end of the window. Each of these contributions depend on the input signal $(u(t), t < 0$ for the begin transient; $u(N-t), t > 0$ for the end transient). Hence, a weak correlation between $T_0^{[m]}(l)$ and $U_0^{[m]}$ will exist. It is shown [9] that this results eventually in a systematic error contribution that can be bounded by

$$\lim_{M\to\infty} \hat{G}^M_{\text{Rect}} = \frac{E\left\{\bar{U}_0^{[m]}T_0^{[m]}\right\}}{E\left\{U_0^{[m]}\bar{U}_0^{[m]}\right\}} = O(N^{-1}) \tag{14}$$

at all excited frequencies $(E\{U_0^{[m]}(l)\bar{U}_0^{[m]}(l)\} \neq 0)$ and

$$\lim_{M\to\infty} \hat{G}^M_{\text{Rect}}(l) = G_0(l) + O(N^{-1}). \tag{15}$$

b) Random errors: In the absence of disturbing noise, the variance of the error term is bounded by an $O(M^{-1}N^{-1})$ [9] and, hence, the random errors dominate the systematic errors for a wide range of choices of M and N.

2) Hanning Window: The errors for the rectangular window are completely due to the leakage term $T_0(l)/U_0(l)$. Applying a Hanning window results eventually in

$$\hat{G}_{\text{Hann}}(l) = \frac{2Y_0(l) - Y_0(l+1) - Y_0(l-1)}{2U_0(l) - U_0(l+1) - U_0(l-1)}. \tag{16}$$

As noted previously, the Hanning window can be considered as a second-order difference that reduces the impact of the transient since T is a smooth function of the frequency.

Define $\Delta = f_s/N$. Using the smoothness of G_0 and T_0, we have

$$\begin{aligned}G_0(l \pm 1) &= G_0 \pm G_0^{(1)}\Delta + G_0^{(2)}\frac{\Delta^2}{2} + O(N^{-3}) \\ T_0(l \pm 1) &= T_0 \pm T_0^{(1)}\Delta + T_0^{(2)}\frac{\Delta^2}{2} \\ &\quad + O(N^{-3})O(N^{-1})\end{aligned} \tag{17}$$

with $X^{(n)}$ the nth derivative of $X(l)$. The last $O(N^{-1})$ is because $T_0^{(3)}$ is an $O(N^{-1})$ [see also (4)]. Substituting (17) in (16) results in

$$\hat{G}_{\text{Hann}}(l) = G_0(l) + e_{1\,\text{Hann}}(l) + e_{2\,\text{Hann}}(l) + O(N^{-3}) \tag{18}$$

with e_1 the leakage and e_2 the interpolation error as follows.

1) Leakage error e_1: This is the term that remains after double differentiation of the transient

$$\begin{aligned}e_{1\,\text{Hann}}(l) &= \frac{-T_0^{(2)}(l)\Delta^2}{2U_0(l) - U_0(l+1) - U_0(l-1)} \\ &= O(N^{-5/2}).\end{aligned} \tag{19}$$

2) Interpolation error e_2: The double difference combines neighboring lines which leads to an "interpolation error" on G_0

$$\begin{aligned}e_{2\,\text{Hann}}(l) &= -G_0^{(1)}\Delta\frac{U_0(l+1) - U_0(l-1)}{2U_0(l) - U_0(l+1) - U_0(l-1)} \\ &\quad - G_0^{(2)}\frac{\Delta^2}{2}\frac{U_0(l+1) + U_0(l-1)}{2U_0(l) - U_0(l+1) - U_0(l-1)} \\ &= O(N^{-1}) + O(N^{-2}).\end{aligned} \tag{20}$$

From (19) and (20), it turns out that the leakage error $e_{1\,\text{Hann}}$ is reduced to an $O(N^{-5/2})$, but compared with the rectangular window a new 'interpolation' term $e_{2\,\text{Hann}}$ appears which is $O(N^{-1})$. Hence the Hann window reduces the error from an $O(N^{-1/2})$ to an $O(N^{-1})$, and it switches the nature of the dominant error from 'leakage' errors to 'interpolation' errors.

Averaging: Again, an averaging procedure is often used

$$\hat{G}^M_{\text{Hann}}(l) = \frac{\sum_{m=1}^{M} Y^{[m]}_{0\,\text{Hann}}(l)\bar{U}^{[m]}_{0\,\text{Hann}}(l)}{\sum_{m=1}^{M} U^{[m]}_{0\,\text{Hann}}(l)\bar{U}^{[m]}_{0\,\text{Hann}}(l)} \tag{21}$$

leading to the following systematic and random errors.

a) Systematic contributions: It is shown [9] that

$$\begin{aligned}\lim_{M\to\infty} \hat{G}^M_{\text{Hann}}(l) &= G_0(l) + 2G_0^{(1)}\frac{P^{(1)}_{u_0u_0}}{6P_{u_0u_0}}\Delta^2 - G_0^{(2)}\frac{\Delta^2}{6}\\ &= G_0(l) + O(N^{-2})\end{aligned} \tag{22}$$

with $P_{u_0u_0} = E\{|U_0(l)|^2\}$.

Compared to the rectangular window, the systematic errors are reduced from $O(N^{-1})$ to $O(N^{-2})$.

b) Random error: The variance of $\hat{G}^M_{\text{Hann}}(l)$ is dominated by the first term in (20) and equals [9]

$$\text{var}\left(\hat{G}^M_{\text{Hann}}(l)\right) = \frac{\left|G_0^{(1)}(l)\Delta\right|}{3M} = O(M^{-1}N^{-2}). \tag{23}$$

C. New Default Window: The Diff Window

The Hanning window reduces the leakage effects on the FRF measurements to $O(N^{-1})$. This error reduction is obtained due to a shift of the nature of the errors from "leakage" (e_1) errors to "interpolation" (e_2) errors. The latter one grow with the width of the interpolation interval which is two bins (three lines) for the Hanning window. An alternative window with a smaller width should allow for a better balancing between the leakage and interpolation errors. This idea is elaborated in Section III-C1.

1) New Window: An alternative for the three-lines second-order difference of the Hanning window is to make only a first-order difference of the spectra that combines only two lines

$$\hat{G}_{\text{Diff}}\left(l+\frac{1}{2}\right) = \frac{Y_0(l+1) - Y_0(l)}{U_0(l+1) - U_0(l)} = \frac{Y_{0\,\text{Diff}}(l)}{U_{0\,\text{Diff}}(l)}$$

and

$$\hat{G}^M_{\text{Diff}}\left(l+\frac{1}{2}\right) = \frac{\sum_{m=1}^{M} Y^{[m]}_{0\,\text{Diff}}(l)\bar{U}^{[m]}_{0\,\text{Diff}}(l)}{\sum_{m=1}^{M} U^{[m]}_{0\,\text{Diff}}(l)\bar{U}^{[m]}_{0\,\text{Diff}}(l)}. \tag{24}$$

Applying again the Taylor-series representation (17), but this time around $l + (1/2)$, results in

$$\begin{aligned}\hat{G}_{\text{Diff}}\left(l+\frac{1}{2}\right) = G_0\left(l+\frac{1}{2}\right) + e_{1\,\text{Diff}}\left(l+\frac{1}{2}\right)\\ + e_{2\,\text{Diff}}\left(l+\frac{1}{2}\right)\end{aligned} \tag{25}$$

with leakage error

$$\begin{aligned}e_{1\,\text{Diff}}\left(l+\frac{1}{2}\right) &= T_0^{(1)}\left(l+\frac{1}{2}\right)\Delta\frac{1}{U_0(l+1) - U_0(l)}\\ &= O(N^{-3/2})\end{aligned} \tag{26}$$

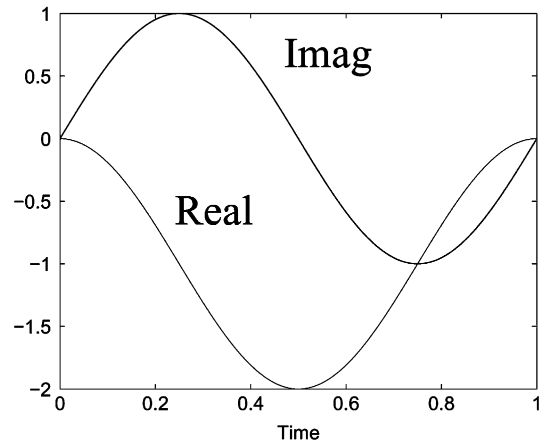

Fig. 1. Real and imaginary part of the complex window corresponding to the first-order difference operation.

and interpolation error

$$\begin{aligned}e_{2\,\text{Diff}}\left(l+\frac{1}{2}\right) &= G_0^{(1)}\left(l+\frac{1}{2}\right)\frac{U_0(l+1)+U_0(l)}{U_0(l+1)-U_0(l)}\frac{\Delta}{2}\\ &\quad + G_0^{(2)}\left(l+\frac{1}{2}\right)\frac{\Delta^2}{8}\\ &= O(N^{-1}) + O(N^{-2}).\end{aligned} \tag{27}$$

$e_{2\,\text{Diff}}$ is reduced with respect to $e_{2\,\text{Hann}}$ by working around the middle frequency $l + (1/2)$. In that case, an approximation is made over only half a bin to the left and to the right instead of a full bin for the Hanning window. The leakage error increased to $O(N^{-3/2})$, but this is not important because it is not the dominating error for N sufficiently large.

a) Systematic error: It is shown that [9]

$$\lim_{M\to\infty} \hat{G}^M_{\text{Diff}}\left(l+\frac{1}{2}\right) = G_0\left(l+\frac{1}{2}\right) + O(N^{-2}) \tag{28}$$

which is of the same order as the Hanning window.

b) Random error: The variance becomes [9]

$$\begin{aligned}\text{var}\left(\hat{G}^M_{\text{Diff}}\left(l+\frac{1}{2}\right)\right) &= \frac{\left|G_0^{(1)}\left(l+\frac{1}{2}\right)\Delta\right|^2}{4M}\\ &= O(M^{-1}N^{-2}).\end{aligned} \tag{29}$$

The variance is slightly reduced (−1.25 dB) compared to the Hanning window [see (23)]. This allows to reduce the measurement time with 25% for the same level of variance of the leakage error on the measured FRF.

2) Time-Domain Interpretation: Making the difference over two neighboring frequencies can be interpreted as applying a complex window in the time domain (Fig. 1)

$$\begin{aligned}&w(k) = e^{j\frac{2\pi}{N}k} - 1 = 2je^{j\frac{\pi}{N}k}\sin\frac{\pi}{N}k, \qquad \text{for } 0 \le k \le N-1\\ &w(k) = 0, \qquad \text{elsewhere.}\end{aligned} \tag{30}$$

From (30), it follows immediately that the spectrum of the diff window equals the spectrum of a half-sine window within a frequency shift of half a bin. It is this frequency shift that allows for the very simple expression of the window in the frequency

TABLE I
COMPARISON OF THE RECTANGULAR, HANNING, AND DIFF WINDOW

window	leak. error e_1	interp. error e_2	syst. error ($M \to \infty$)	variance
w_{Rect}	$O(N^{-1/2})$	0	$O(N^{-1})$	$O(M^{-1}N^{-1})$
w_{Hann}	$O(N^{-5/2})$	$O(N^{-1})$	$O(N^{-2})$	$O(M^{-1}N^{-2})$
w_{Diff}	$O(N^{-3/2})$	$O(N^{-1})$	$O(N^{-2})$	$O(M^{-1}N^{-2})$

domain (making the difference over two neighboring frequencies of the DFT spectrum obtained with a rectangular window). It can also be noted that the properties of the diff window and the half-sine window will be the same, but the numerical calculation/interpretation in the frequency domain is simplified. This also allows the interested reader to position the alternative diff window against other windows that are studied in the literature.

D. Conclusion

In Table I, all the results of the previous discussions are collected. It is seen that for FRF measurements, the Hanning window is superior to the rectangular window, while the diff window even does a little bit better on all aspects studied. The diff window can replace the Hanning window as default choice in FRF measurements.

IV. Alternative Interpretation of the Hanning Window: A Taylor-Series Approach

Consider the Taylor approximation in (17), with order 0 for G_0, and order 1 for T at frequencies $k-1, k, k+1$

$$\begin{aligned} Y_0(k+1) &\approx G_0(k)U_0(k+1)T(k) + \Delta_T \\ Y_0(k) &\approx G_0(k)U_0(k) + T(k) \\ Y_0(k-1) &\approx G_0(k)U_0(k-1) + T(k) - \Delta_T. \end{aligned} \tag{31}$$

Solving (31) for $G_0(k)$ leads to

$$\hat{G}(k) = \frac{2Y_0(k) - Y_0(k+1) - Y_0(k-1)}{2U_0(k) - U_0(k+1) - U_0(k-)} \tag{32}$$

which is nothing than the estimate $\hat{G}_{\mathrm{Hann}}(k)$ found by applying the Hanning window. This clearly indicates that the Hanning window is a nonoptimal choice. It puts to much emphasis on the elimination of the transient. To eliminate this unbalance, also G_0 should be expanded to the first order, leading to

$$\begin{aligned} G_0(l+\Delta) &= G_0(l) + g_1\Delta + g_2\Delta^2 + O(\Delta^3) \\ T_0(l+\Delta) &= T_0(l) + t_1\Delta + O(N^{-1})O(\Delta^2). \end{aligned} \tag{33}$$

Substituting this result in

$$Y_0(l) = G_0(l)U_0(l) + T(l), \quad \text{for} \quad l = k-2, k-1, k, k+1, k+2 \tag{34}$$

would lead to a set of five complex equations with five complex unknowns. However, the additional gain would be lost because now five lines have to be combined. The alternative is to consider again only three lines $l = k-1, k, k+1$, but this for $M \geq 4$ realizations. This leads to $3 + 2M$ unknowns ($G_0(k), g_1(k), g_2(k)$ and M times $T_0(k), t_1(k)$). The set of $3M$ equations with $2M + 3$ unknowns is solved in least-squares sense. This leads eventually to

$$\hat{G}_{\mathrm{Tayl}}(k) = G_0(k) + O(N^{-5/2}). \tag{35}$$

V. Noise Sensitivity

The analysis in the previous sections was made assuming that the disturbing noise equals zero. The three windows (rectangular, Hanning, diff) resulted eventually in the same type of estimates

$$\hat{G} = \frac{Z}{X_0} = \frac{Z_0 + N_Z}{X_0} \tag{36}$$

where Z and X are defined in (12), (16), and (24). For multiple measurements, $Z^{[l]}$ and $X_0^{[l]}, l = 1, \ldots, M$ are available, and the H_1 averaging technique is used [1]

$$\hat{G}^M = \sum_{m=1}^{M} Z^l \bar{X}_0^l \Big/ \sum_{m=1}^{M} X_0^l \bar{X}_0^l. \tag{37}$$

The variance for $\hat{G}^M_{\mathrm{Rect}}, \hat{G}^M_{\mathrm{Diff}}$, and $\hat{G}^M_{\mathrm{Hann}}$ is approximately given by

$$\sigma_G^2 = \frac{\sigma_{N_Z}^2}{ME\{|X|^2\}}. \tag{38}$$

This shows that the noise sensitivity of all these estimators is the same and the variance due to the disturbing noise is

$$\sigma_G^2 = \frac{\sigma_V^2}{ME\{|U_0|^2\}} \tag{39}$$

with $E\{\ \}$ the expected value taken over the successive realizations of the input signal.

For small $M, (1/M)\sum_{l=1}^{M} |U_0^{[l]}(k)|^2$ can be significantly different from $E\{|U_0|^2\}$. At some frequencies, large drops in the realized power spectrum appear, jeopardizing the FRF measurement completely. Therefore, it is advised to choose M large enough to avoid these dips [8].

Also, for G_{Tayl}, an explicit noise analysis can be made using the classical results of least-squares estimates. However, the reader should be aware that the number of unknown parameters grows with the number M of averaged experiments because for each experiment additional transient parameters are estimated. This leads eventually to an inefficiency term, such that

$$\sigma_{G_{\mathrm{Tayl}}}^2 = \sigma_G^2 + g(M) \tag{40}$$

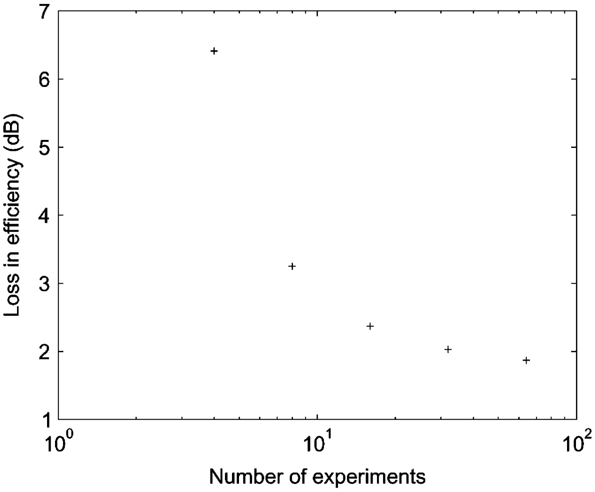

Fig. 2. Efficiency loss of the Taylor method as a function of the number of averaged experiments. The limit value for $M \to \infty$ is 1.24 dB.

with σ_G^2 the variance of the windowing methods. Analytically, it is found that (Appendix 1)

$$\lim_{M\to\infty} g(M) = \sigma_G^2/3. \tag{41}$$

For finite values of M, it can be obtained through simulations, and it is plotted in Fig. 2. As can be seen, the loss is about 6 dB for $M = 4$. Hence, the Taylor method can only be used if the leakage errors dominate the disturbing noise errors.

VI. Illustration

A discrete-time system is excited with white Gaussian noise. $M = 64$ experiments of 8192 points are processed, such that 1024 frequency points in the frequency band of interest are available. First, a noise-free simulation is repeated 1000 times. No disturbing noise is added $(v(t) = 0)$ in order to be able to emphasize the effects that are described in this paper. The mean and the standard deviation for the three FRF estimators are calculated and the results are shown in Fig. 3. For all cases, the random errors dominate. Note that the new "diff" window does slightly better than the Hanning window as was expected from the theory. It can also be noted that the "Taylor" method has a superior behavior.

Next 100 simulations with $M = 16$ are made with white disturbing noise added to the output. The results are shown in Fig. 4. In this figure, it is clearly seen that at the resonances, where the leakage error dominates, the Taylor method is still superior. Outside these frequency bands, it can be seen that the errors for the window methods become about 2 dB smaller than for the Taylor method, which is due to a lower noise sensitivity as explained previously.

VII. Conclusion

In this paper, an analysis of the windowing/leakage effects on FRF measurements is made. It turns out that the leakage errors in FRF measurements have a highly structured nature that can be used to reduce their impact. The arguments used in window analysis for spectral analysis applications cannot be unaltered transferred to FRF measurements. Replacing the rectangular window by a Hanning shifts the nature of the error from leakage to interpolation. It turns out that an alternative "diff" window can be proposed with slightly better properties. It allows a reduction of the measurement time with 25% if leakage errors dominate. If the output noise is the dominating error source,

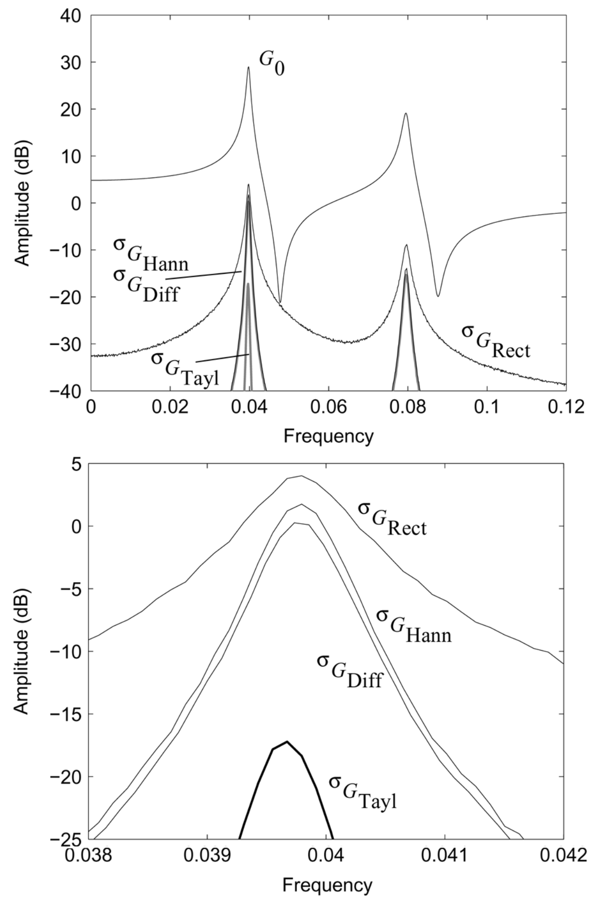

Fig. 3. Comparison of the standard deviation of the errors on four FRF estimators: $G_{\mathrm{Hann}}, G_{\mathrm{Diff}}, G_{\mathrm{Rect}},$ and G_{Tayl}, together with the exact valueG_0 of the FRF. Top: global view. Bottom: zoom around the first resonance frequency.

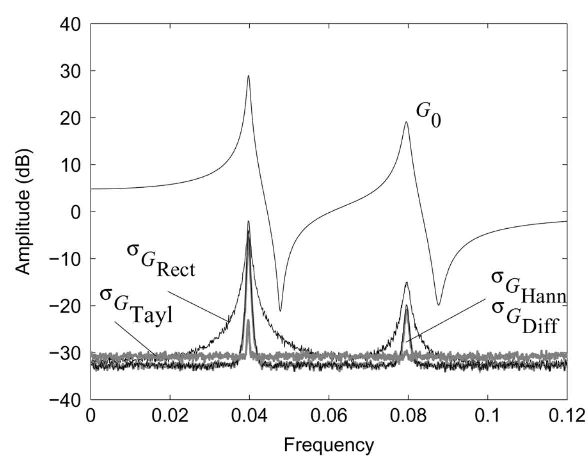

Fig. 4. Comparison of the standard deviation of the errors on four FRF-estimators: $G_{\mathrm{Hann}}, G_{\mathrm{Diff}}, G_{\mathrm{Rect}},$ and G_{Tayl}, together with the exact value of the FRF.

both windows have the same disturbing noise sensitivity. Eventually, a Taylor-series interpretation of the Hanning window is made. This leads to an improved FRF estimate with significantly smaller errors.

APPENDIX
NOISE ANALYSIS OF THE TAYLOR METHOD

To calculate the covariance matrix starting from a group of M repeated experiments, the set of (31) is written in matrix form. Define

$$Y_{\mathrm{All}}^T = \Big[Y^{[1]}(k-1), Y^{[1]}(k), Y^{[1]}(k+1), \ldots Y^{[M]}(k-1), Y^{[M]}(k), Y^{[M]}(k+1)\Big] \quad (42)$$

with X^T the transpose of X, and

$$Z_{\mathrm{All}}^T = \Big[G(k), g_1(k), g_2(k), T^{[1]}(k), t_1^{[1]}(k) \ldots T^{[M]}(k), t_1^{[M]}(k)\Big]. \quad (43)$$

Then, (31) becomes

$$Y_{\mathrm{All}} = K Z_{\mathrm{All}}, \qquad \text{with} \quad K \in C^{3M\times(3+2M)}. \quad (44)$$

The covariance matrix of the least-squares solution of (44) is given by

$$C = \sigma_V^2(k)(K^H K)^{-1} \quad (45)$$

because after a DFT, for N sufficiently large, the noise is asymptotically uncorrelated distributed, and varies only slowly over the frequencies if the noise spectrum is smooth.

The matrix inversion in (45) can be calculated by considering

$$K^H K = \begin{bmatrix} c_{11} & c_{12} \\ c_{12}^H & c_{22} \end{bmatrix} \quad (K^H K)^{-1} = \begin{bmatrix} d_{11} & d_{12} \\ d_{12}^H & d_{22} \end{bmatrix} \quad (46)$$

with $c_{11} \in C^{3\times3}, c_{12} \in C^{3\times2M}$, and $c_{22} \in C^{2M\times2M}$. The result of interest is in

$$d_{11} = c_{11}^{-1} + c_{11}^{-1} c_{12} c_{22}^{-1} c_{12} c_{11}^{-1}. \quad (47)$$

$c_{12}c_{22}^{-1}c_{12}$ and c_{11} become asymptotically for $M \to \infty$

$$M|U_0(k)|^2 \begin{bmatrix} 2 & 0 & 5/3 \\ 0 & 5/3 & 0 \\ 5/3 & 0 & 5/3 \end{bmatrix}$$

and

$$M|U_0(k)|^2 \begin{bmatrix} 3 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}. \quad (48)$$

c_{22} converges to a diagonal matrix with a repetition of 3, 2 on its main diagonal. Combining all these results leads to (40) and (41).

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