Adapted from AIMA slides

Logic: automated reasoning, provers

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Outline

- Truth and proofs
- Inference rules and theorem proving
  - forward chaining
  - backward chaining
  - Resolution
    - Resolution as elementary inference step
    - Resolution as general inference method
    - Conversion to conjuctive normal form (CNF)
    - Resolution heuristics
- Exercises

Reminder: Knowledge bases

- Knowledge base = set of sentences in a formal language
- Declarative approach to building an agent (or other system):
  - Tell it what it needs to know
  -
- Then it can Ask itself what to do - answers should follow from the KB
-
- Agents can be viewed at the knowledge level
  i.e., what they know, regardless of how implemented
- Or at the implementation level
  - i.e., data structures in KB and algorithms that manipulate them
  -

Reminder: Models

- Logicians typically think in terms of models, which are formally structured worlds with respect to which truth can be evaluated
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- We say *m* is a model of a sentence
- *M(α)* is the set of all models of α
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- Then KB $\models$ α iff *M(KB)* $\subseteq$ *M*(α)
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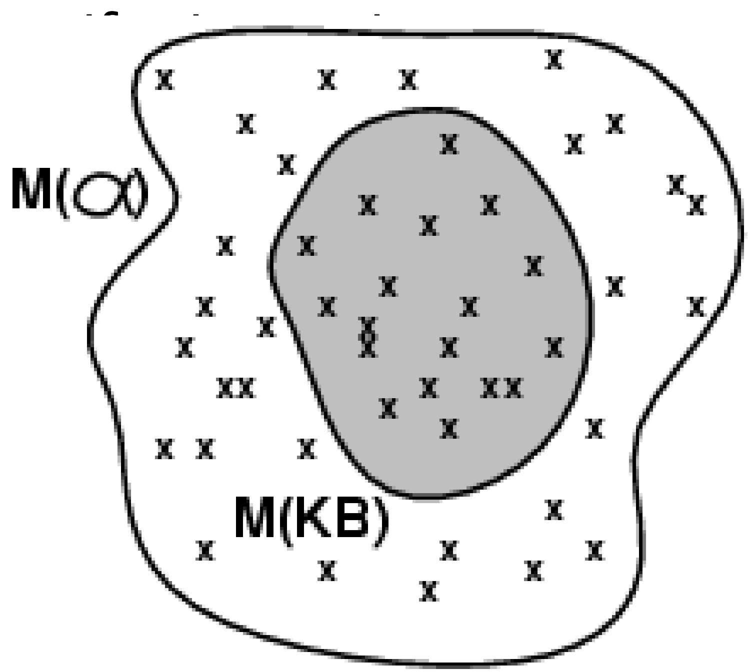
  - E.g. *KB* = Giants won and Reds won α = Giants won
  -



Reminder: truth vs. proof

- Soundness: *i* is sound if whenever $KB \vdash_i \alpha$, it is also true that $KB \models \alpha$
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- Completeness: *i* is complete if whenever $KB \models \alpha$, it is also true that $KB \vdash_i \alpha$
-
- Preview: we will define a logic (first-order logic) which is expressive enough to say almost anything of interest, and for which there exists a sound and complete inference procedure.
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- That is, the procedure will answer any question whose answer follows from what is known by the *KB*.
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Forward and backward chaining

- Horn Form (restricted)
  KB = conjunction of Horn clauses
  - Horn clause =
    - proposition symbol; or
    - (conjunction of symbols) ⇒ symbol
  - E.g., $C \wedge (B \Rightarrow A) \wedge (C \wedge D \Rightarrow B)$
- Modus Ponens (for Horn Form): complete for Horn KBs

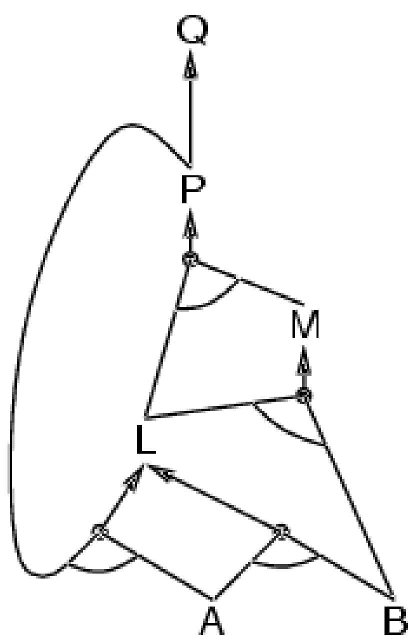
$$\frac{\alpha_1, \ldots, \alpha_n, \qquad \alpha_1 \wedge \ldots \wedge \alpha_n \Rightarrow \beta}{\beta}$$

- Can be used with forward chaining or backward chaining.
- These algorithms are very natural and run in linear time

Forward chaining

- Idea: fire any rule whose premises are satisfied in the *KB*,
  - add its conclusion to the *KB*, until query is found

$$P \Rightarrow Q$$
$$L \wedge M \Rightarrow P$$
$$B \wedge L \Rightarrow M$$
$$A \wedge P \Rightarrow L$$
$$A \wedge B \Rightarrow L$$
$$A$$
$$B$$



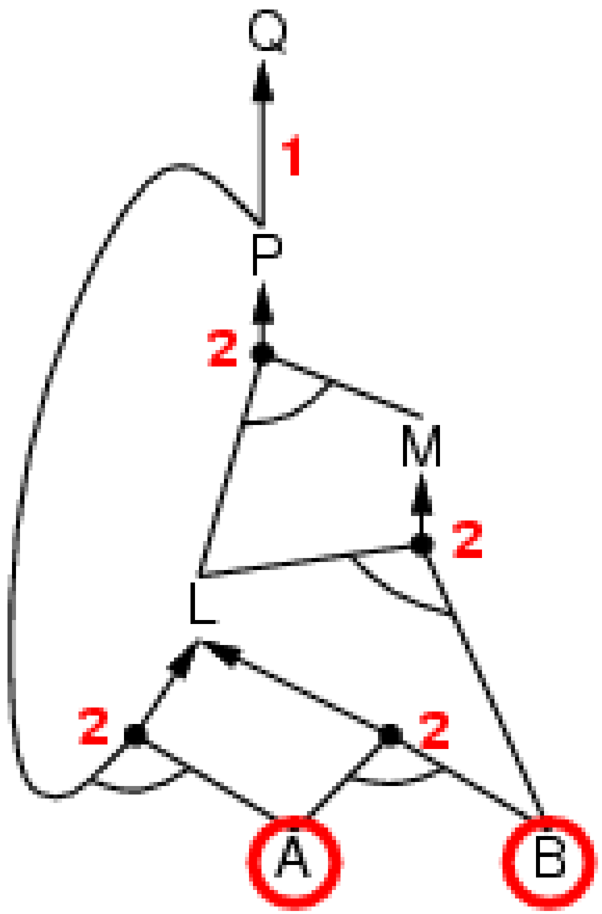
Forward chaining algorithm

```
function PL-FC-ENTAILS?(KB, q) returns true or false
  local variables: count, a table, indexed by clause, initially the number of premises
                   inferred, a table, indexed by symbol, each entry initially false
                   agenda, a list of symbols, initially the symbols known to be true

  while agenda is not empty do
      p ← POP(agenda)
      unless inferred[p] do
          inferred[p] ← true
          for each Horn clause c in whose premise p appears do
              decrement count[c]
              if count[c] = 0 then do
                  if HEAD[c] = q then return true
                  PUSH(HEAD[c], agenda)
  return false
```

Forward chaining is sound and complete for Horn KB

Forward chaining example



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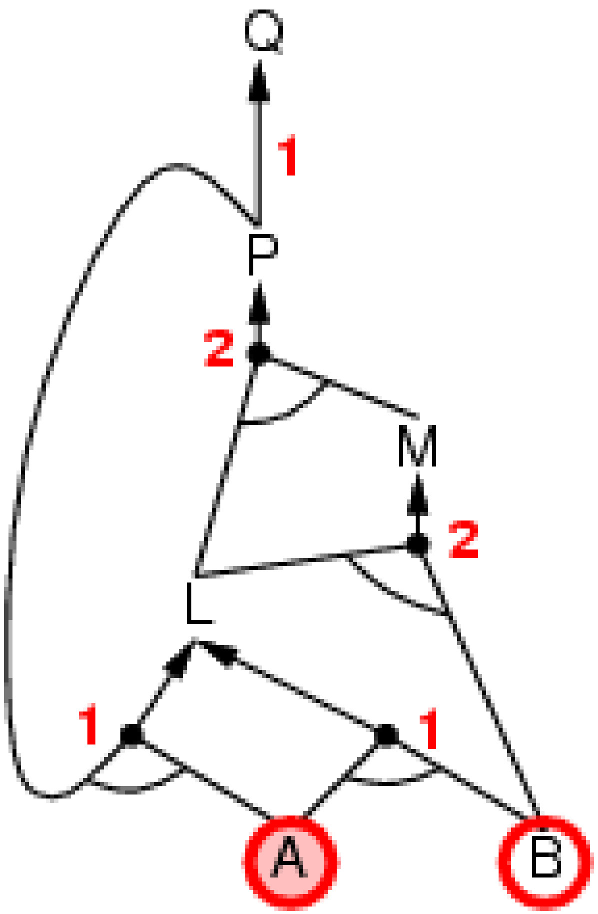

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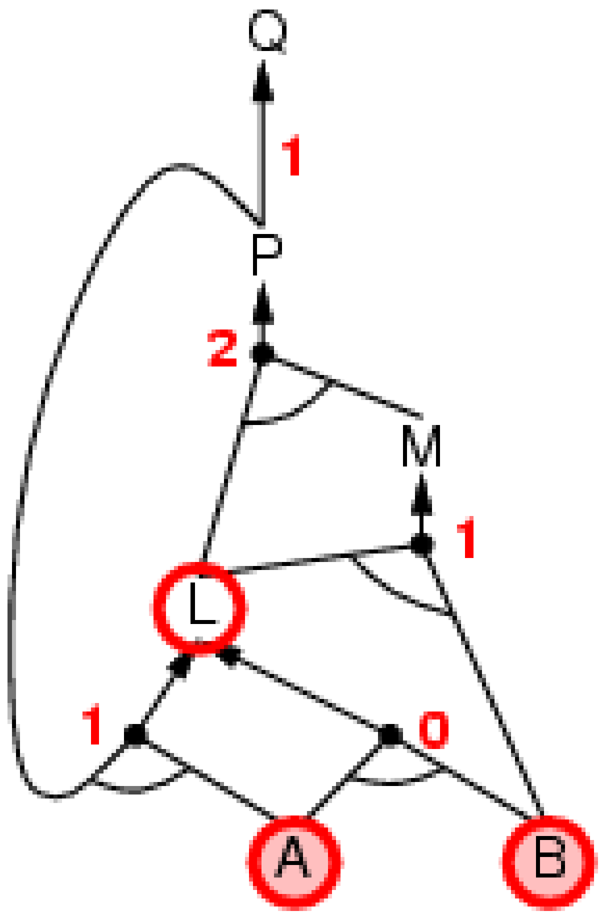

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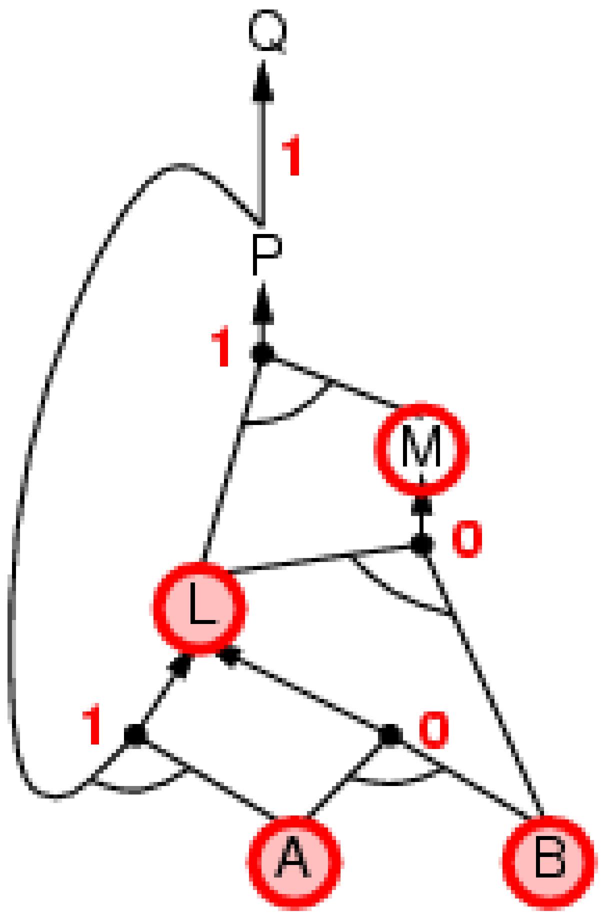

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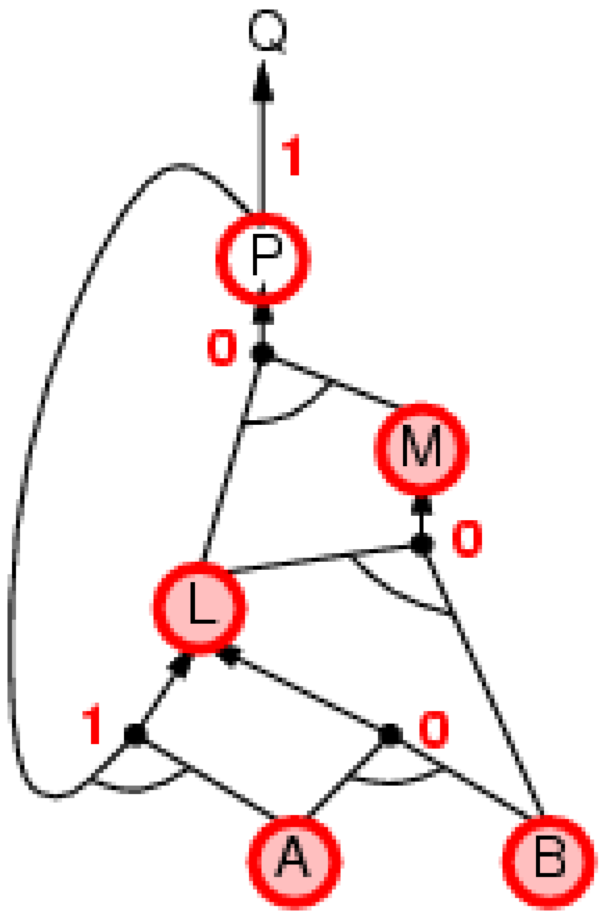

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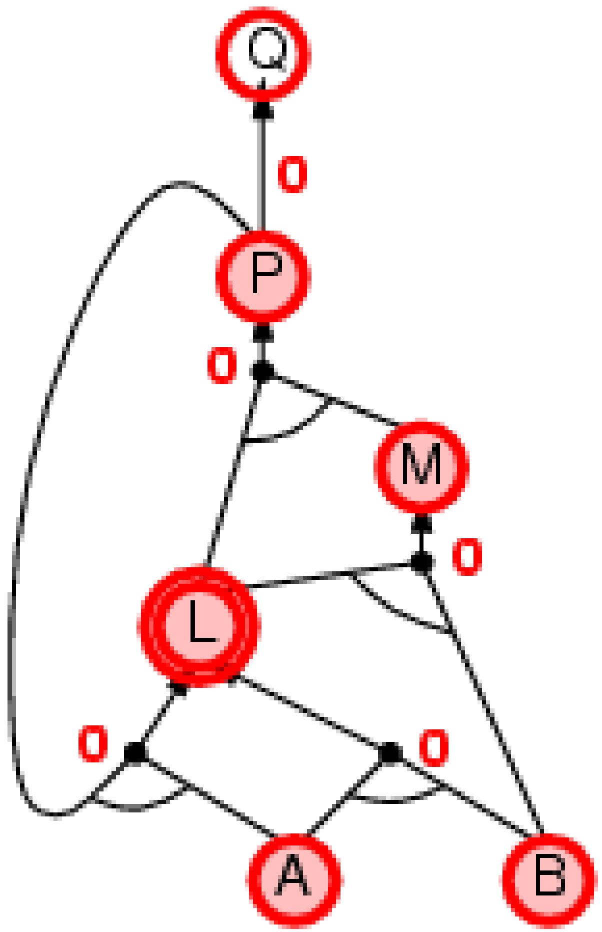

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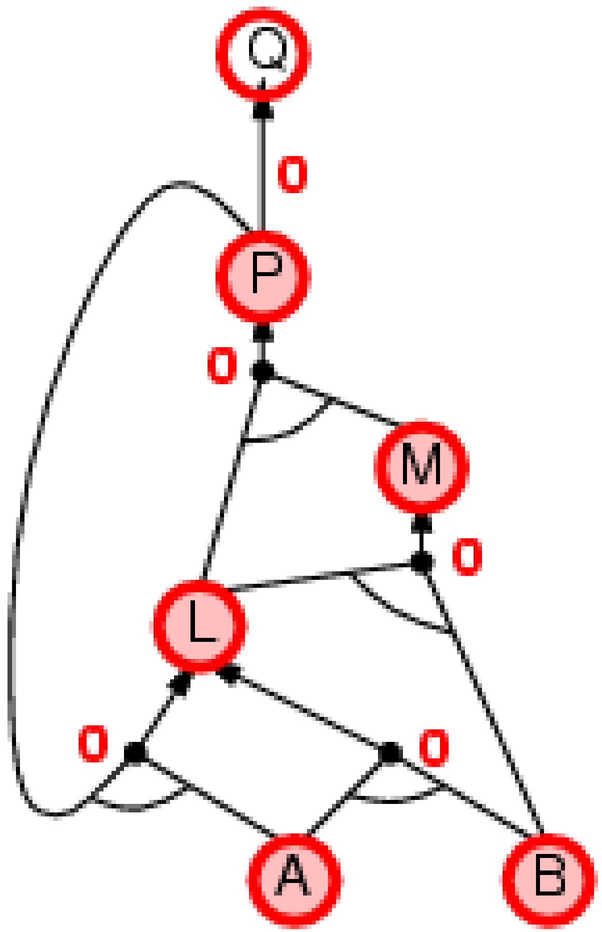

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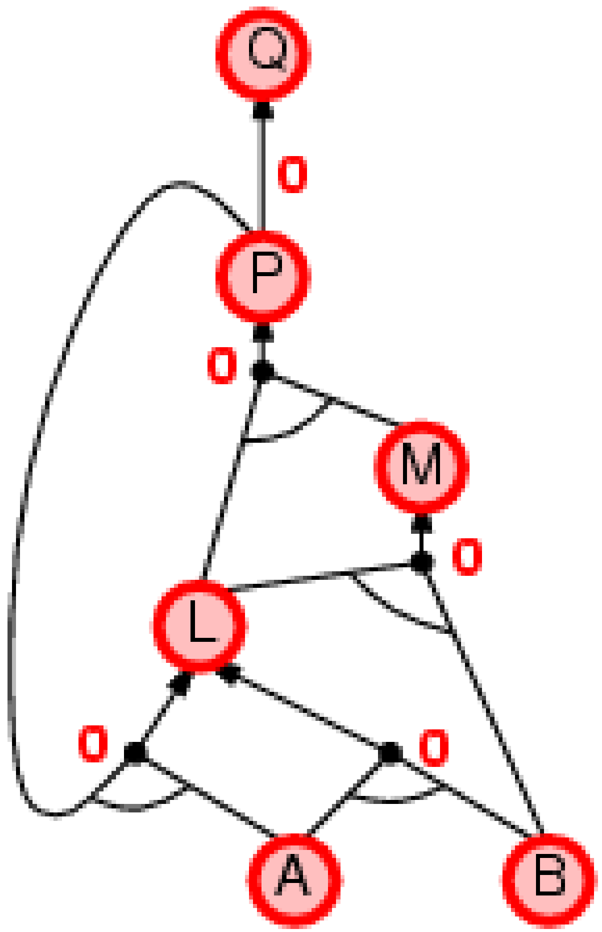


Backward chaining

Idea: work backwards from the query *q*:

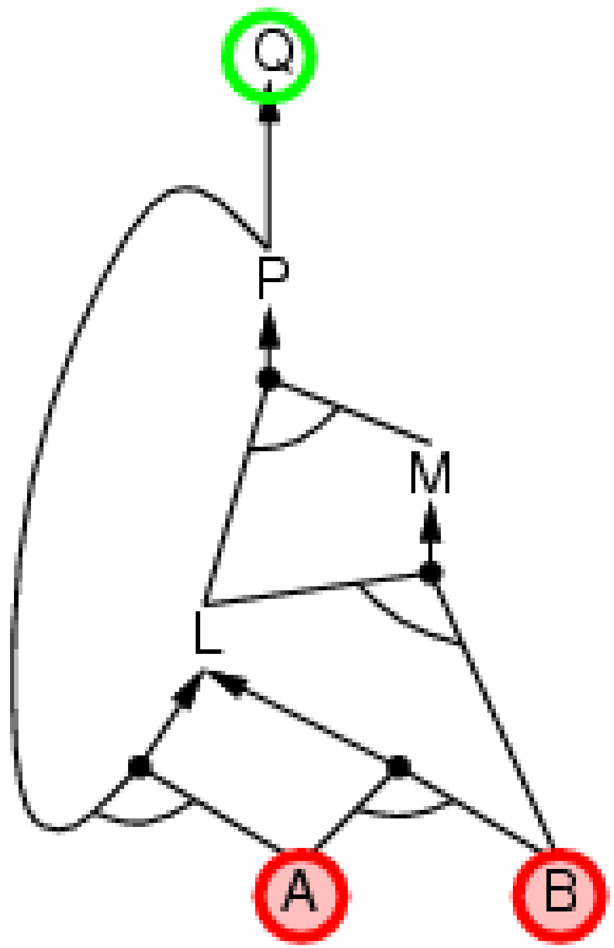
to prove *q* by BC,
check if *q* is known already, or
prove by BC all premises of some rule concluding *q*

Avoid loops: check if new subgoal is already on the goal stack

Avoid repeated work: check if new subgoal

1. has already been proved true, or
2. has already failed

Backward chaining example



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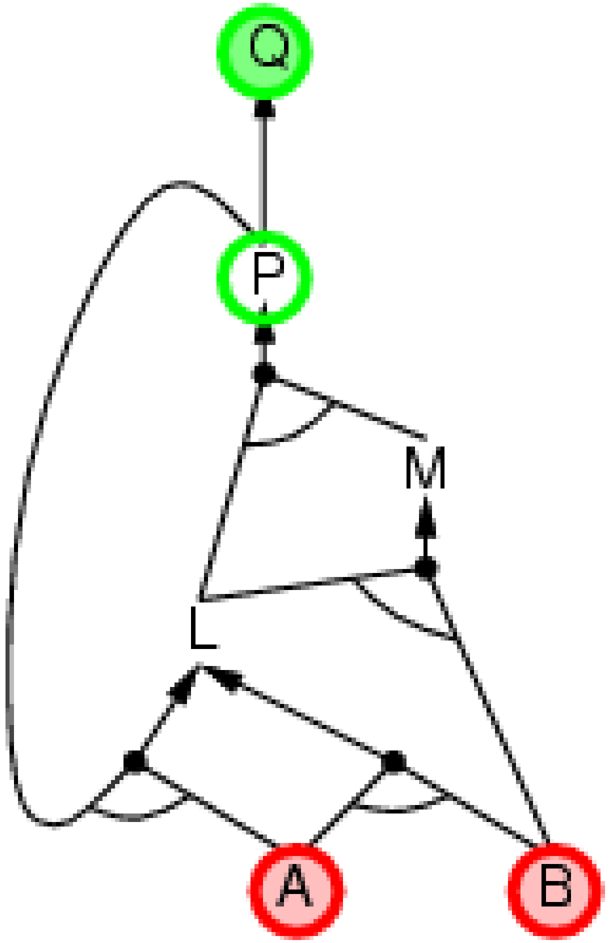

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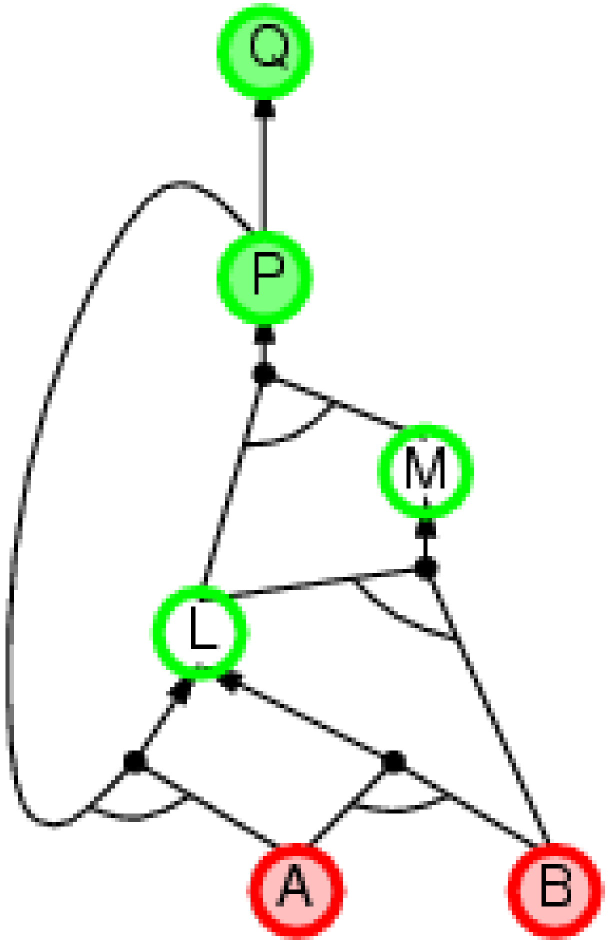

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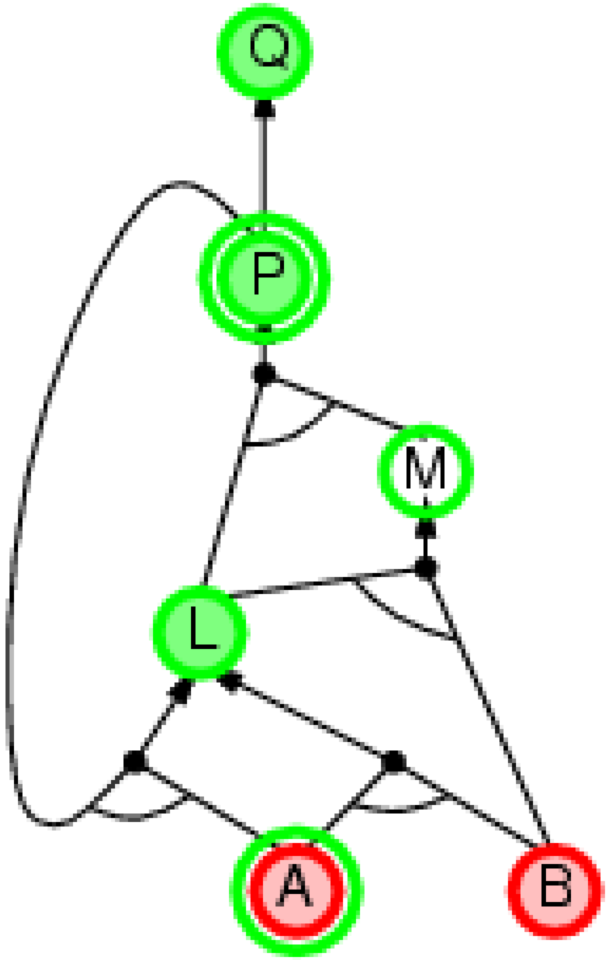

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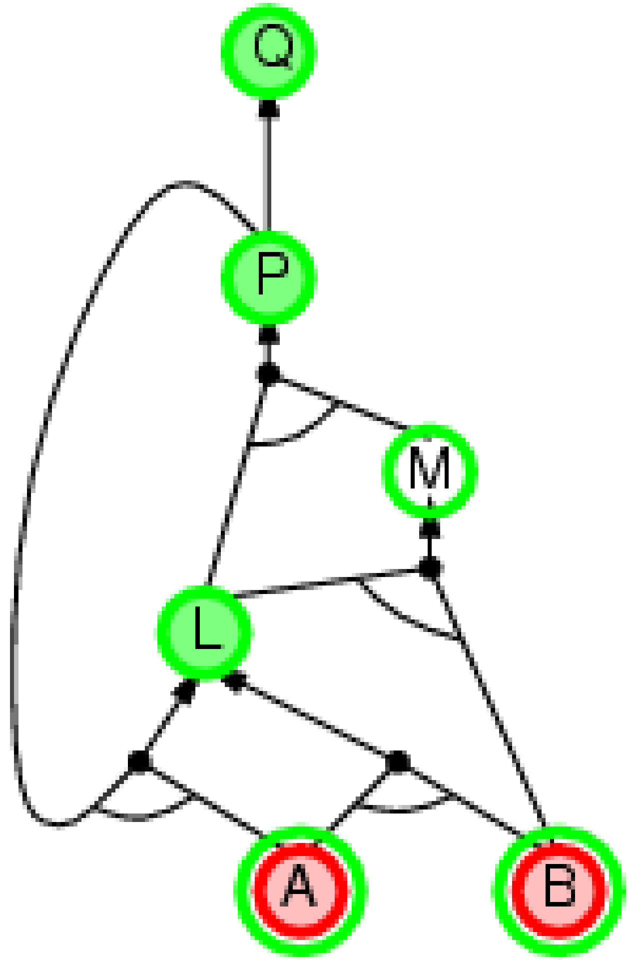

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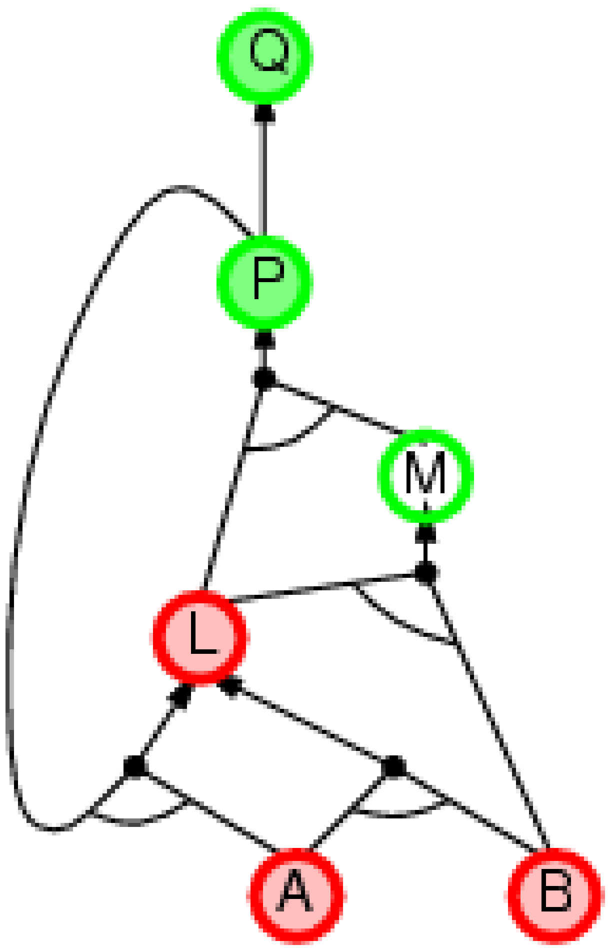

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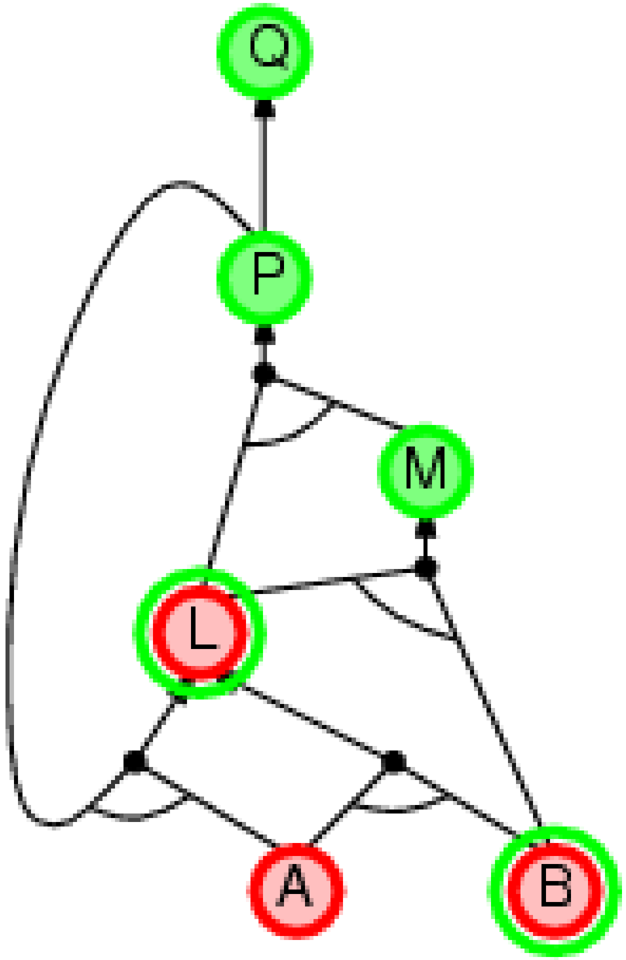

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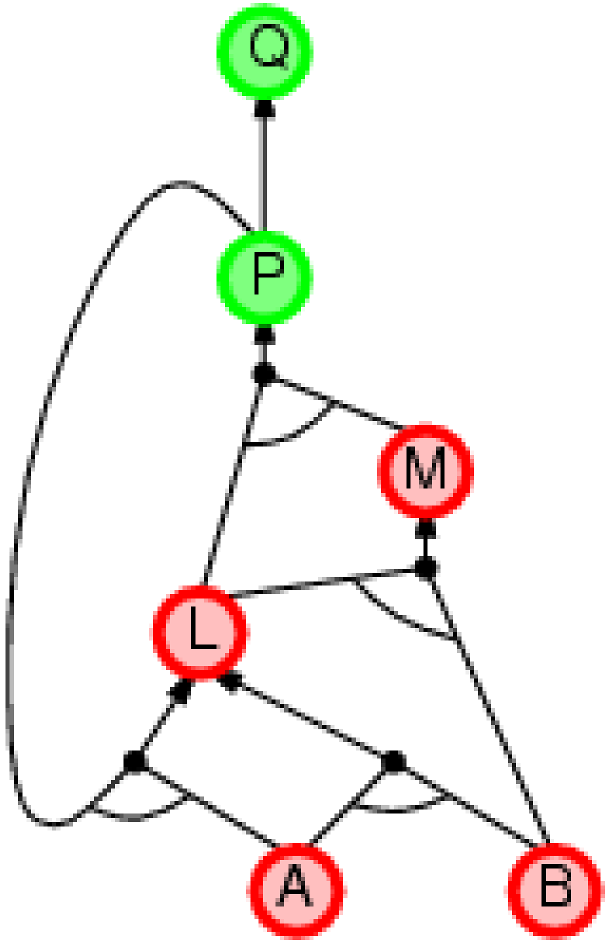

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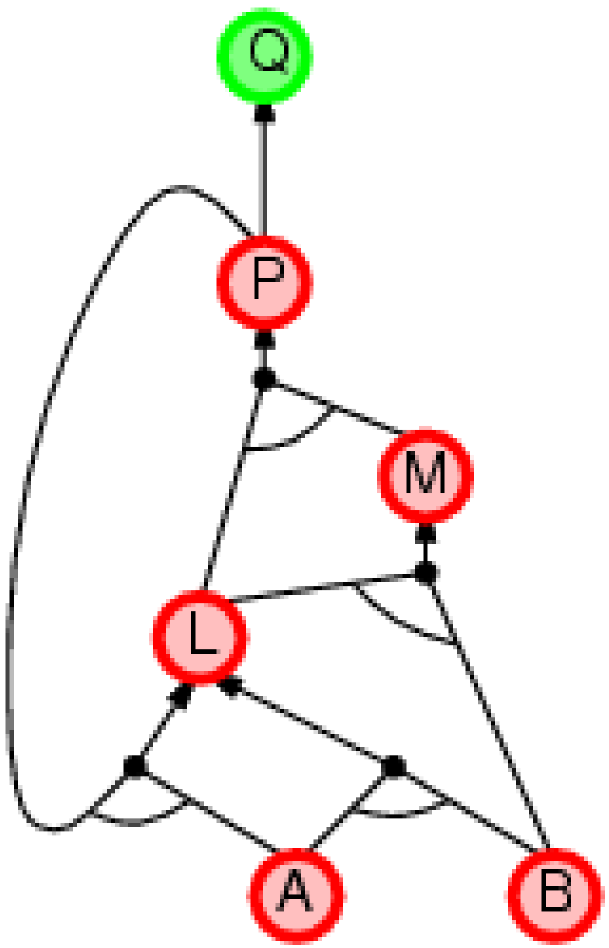

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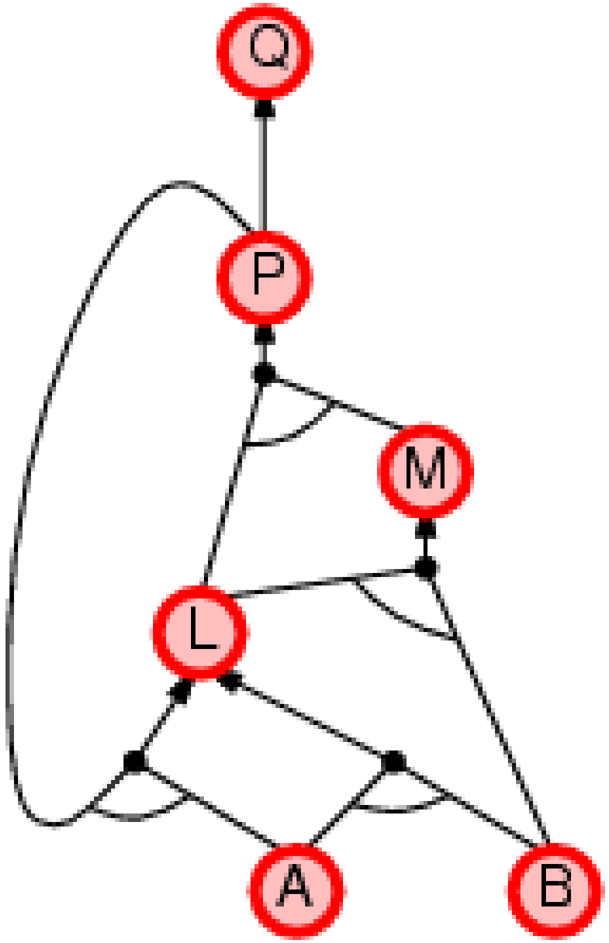

Forward vs. backward chaining

- FC is data-driven, automatic, unconscious processing,
  - e.g., object recognition, routine decisions
  -
- May do lots of work that is irrelevant to the goal
- BC is goal-driven, appropriate for problem-solving,
  - e.g., Where are my keys? How do I get into a PhD program?
- Complexity of BC can be much less than linear in size of KB
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Resolution

Conjunctive Normal Form (CNF)

conjunction of disjunctions of literals

clauses

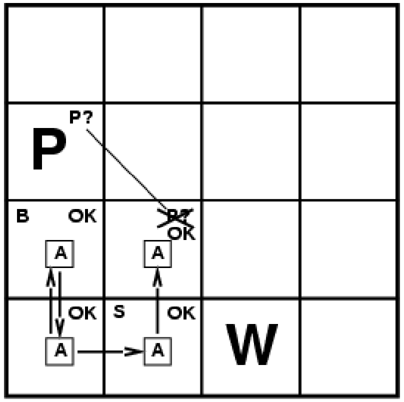
E.g., $(A \vee \neg B) \wedge (B \vee \neg C \vee \neg D)$

- Resolution inference rule (for CNF):
-

$$\frac{l_i \vee \ldots \vee l_k, \qquad m_1 \vee \ldots \vee m_n}{l_i \vee \ldots \vee l_{i-1} \vee l_{i+1} \vee \ldots \vee l_k \vee m_1 \vee \ldots \vee m_{j-1} \vee m_{j+1} \vee \ldots \vee m_n}$$

where l_i and m_j are complementary literals.

E.g., $$\frac{P_{1,3} \vee P_{2,2}, \qquad \neg P_{2,2}}{P_{1,3}}$$



- Resolution is sound and complete for propositional logic

Resolution

Soundness of resolution inference rule:

$$\frac{\begin{array}{c}\neg(\ell_1 \vee \ldots \vee \ell_{i-1} \vee \ell_{i+1} \vee \ldots \vee \ell_k) \Rightarrow \ell_i \\ \neg m_j \Rightarrow (m_1 \vee \ldots \vee m_{j-1} \vee m_{j+1} \vee \ldots \vee m_n)\end{array}}{\neg(\ell_1 \vee \ldots \vee \ell_{i-1} \vee \ell_{i+1} \vee \ldots \vee \ell_k) \Rightarrow (m_1 \vee \ldots \vee m_{j-1} \vee m_{j+1} \vee \ldots \vee m_n)}$$

Conversion to CNF

$B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})\beta$

1. Eliminate $\Leftrightarrow$, replacing $\alpha \Leftrightarrow \beta$ with $(\alpha \Rightarrow \beta)\wedge(\beta \Rightarrow \alpha)$.
2.

   $(B_{1,1} \Rightarrow (P_{1,2} \vee P_{2,1})) \wedge ((P_{1,2} \vee P_{2,1}) \Rightarrow B_{1,1})$

2. Eliminate $\Rightarrow$, replacing $\alpha \Rightarrow \beta$ with $\neg\alpha\vee \beta$.

   $(\neg B_{1,1} \vee P_{1,2} \vee P_{2,1}) \wedge (\neg(P_{1,2} \vee P_{2,1}) \vee B_{1,1})$

3. Move $\neg$ inwards using de Morgan's rules and double-negation:

   $(\neg B_{1,1} \vee P_{1,2} \vee P_{2,1}) \wedge ((\neg P_{1,2} \wedge \neg P_{2,1}) \vee B_{1,1})$

4. Apply distributivity law ($\wedge$ over $\vee$) and flatten:

   $(\neg B_{1,1} \vee P_{1,2} \vee P_{2,1}) \wedge (\neg P_{1,2} \vee B_{1,1}) \wedge (\neg P_{2,1} \vee B_{1,1})$

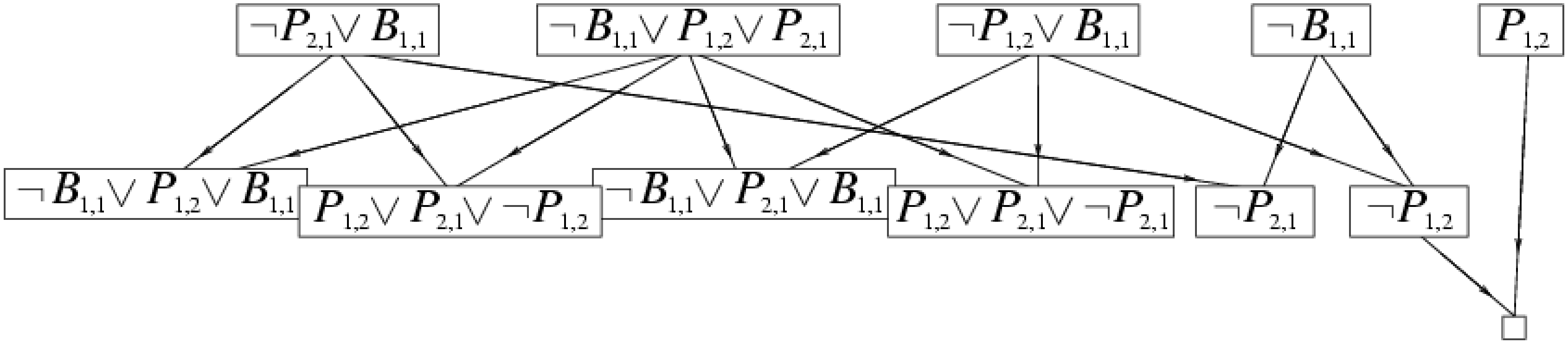
Resolution algorithm

- Proof by contradiction, i.e., show $KB \wedge \neg\alpha$ unsatisfiable

```
function PL-RESOLUTION(KB, α) returns true or false
    clauses ← the set of clauses in the CNF representation of KB ∧ ¬α
    new ← { }
    loop do
        for each Ci, Cj in clauses do
            resolvents ← PL-RESOLVE(Ci, Cj)
            if resolvents contains the empty clause then return true
            new ← new ∪ resolvents
        if new ⊆ clauses then return false
        clauses ← clauses ∪ new
```

Resolution example

- $KB = (B_{1,1} \Leftrightarrow (P_{1,2} \lor P_{2,1})) \land \neg B_{1,1}$ $\alpha = \neg P_{1,2}$



Resolution strategies (heuristics for clause selection)

1. **Unit clause preference**: P, $\neg P \vee [.....] \Rightarrow [.....]$ shorter!

2. **'Set of Support'**

   resolution (a clause from a 'Set of Support‘ and an external clause), rezolvent into 'Set of Support'-ba,

   complete, if clauses not in 'Set of Support‘ are satisfiable

   in practice: 'Set of Support' = the negated question (the rest is assumed to be true)

3. **Input resolution**

   The resolvent in step i. is one of the clause in step i+1 (it starts with the question). **Complete in Horn KBs.**

4. **Linear resolution**

   P and Q can be resolved, if P is in the KB or P is the ancestor of Q in the proof tree. **Complete.**

5. **Pruning**

   Eliminate all rules more specific than in the knowledge base.

Summary

- Truth and proofs
- The „truth-table method” for validity&soundness
- Automated reasoning
  - Forward chaining, Backward chaining
    - linear-time, complete for Horn clauses
  - Resolution
    - Conjunctive normal form (CNF)
    - Inference step
      - Equivalence with if-then forms („transitivity”)
      - Complexity preserving (cf. Modus Ponens)
      - Covers Modus Ponens(!, unit clause)
    - Framework
      - proof by refutation, reductio ad absurdum
      - Heuristics: resolution strategy
    - Complete for propositional logic