Artificial Intelligence: Constraint satisfaction problems

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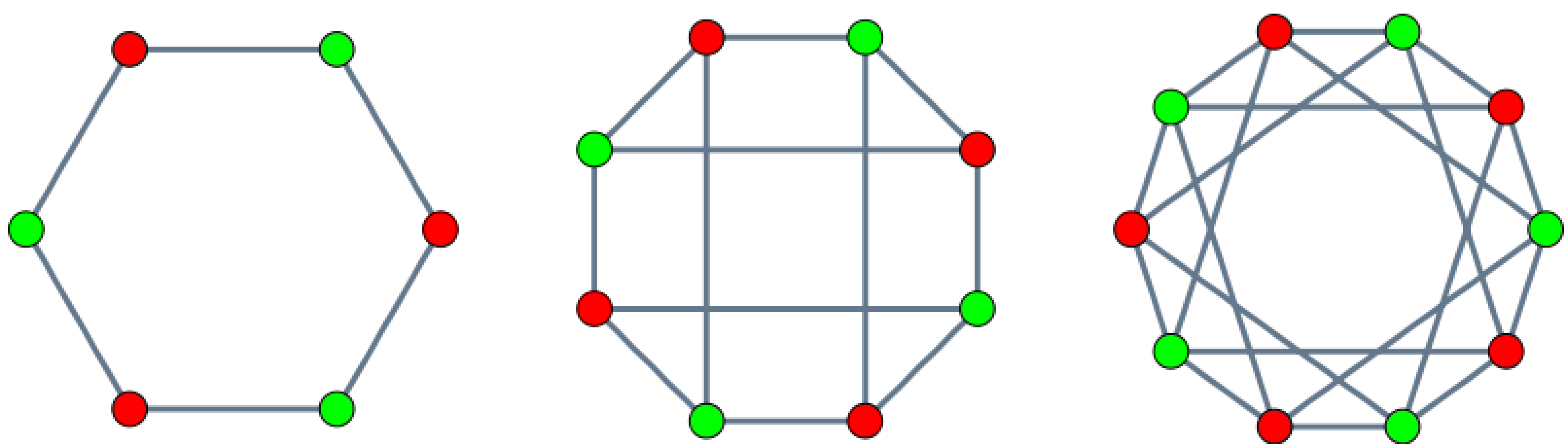
Outline

- Constraint satisfaction problem
- Search in games
- Chess and cognition

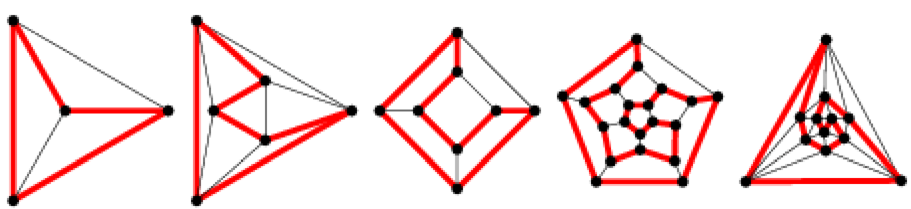
Party: seating arrangements

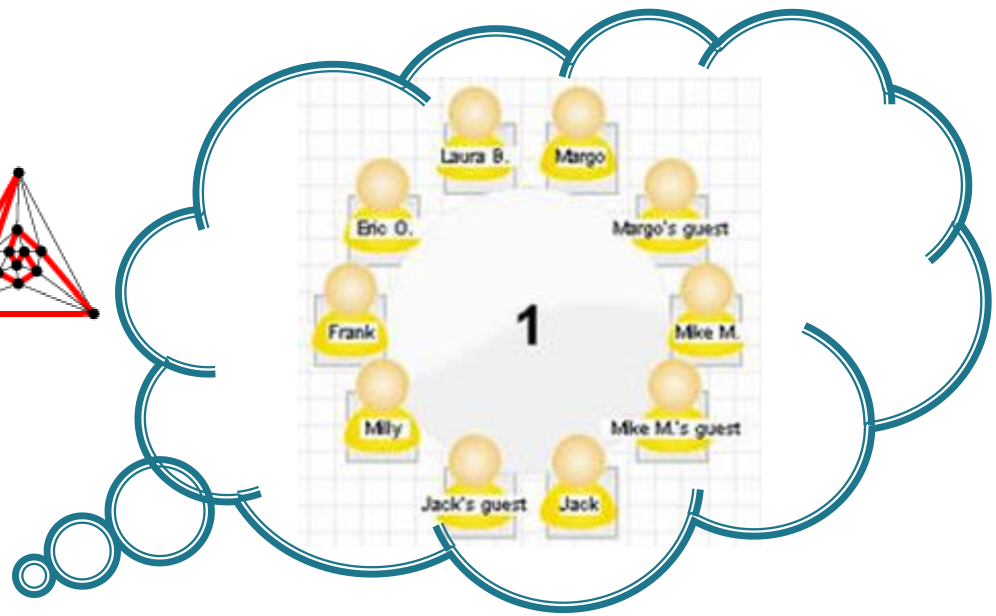
- The ménage problem
  - the number of different ways in which it is possible to seat a set of male–female couples at a dining table so that men and women alternate and nobody sits next to his or her partner.



Seating arrangements: Hamiltonian

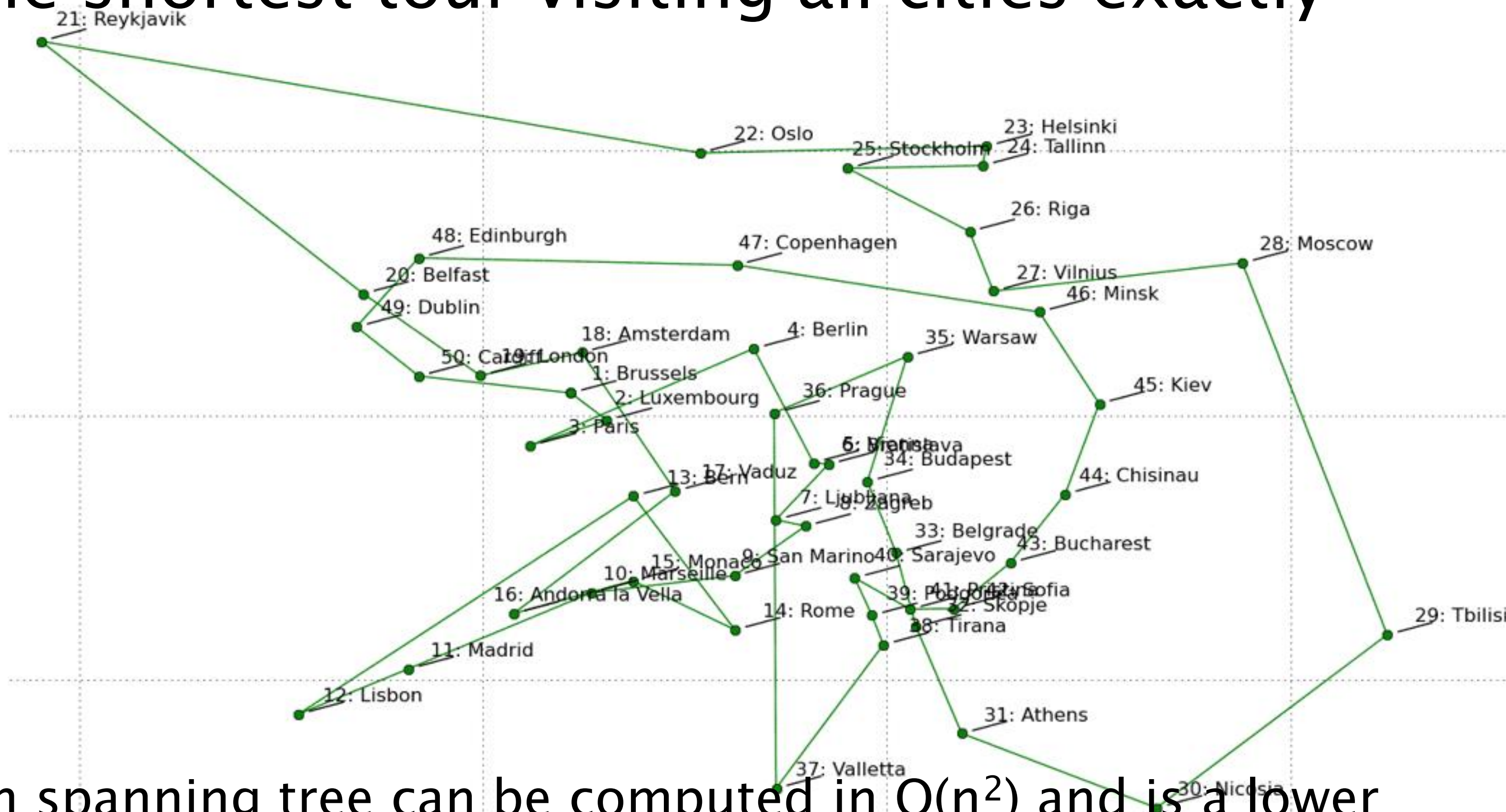
- Sit the guests around a round table with no "incompatible guests" sitting next to each other ?
  - Hamiltonian path/cycle (NP-complete):
    - a path/cycle in a graph that visits each vertex exactly once.
  - Eulerian path/cycle $(<O(E^2))$:
    - a trail/cycle in a graph which visits every edge exactly once.





Travelling sales person problem

- Find the shortest tour visiting all cities exactly once.

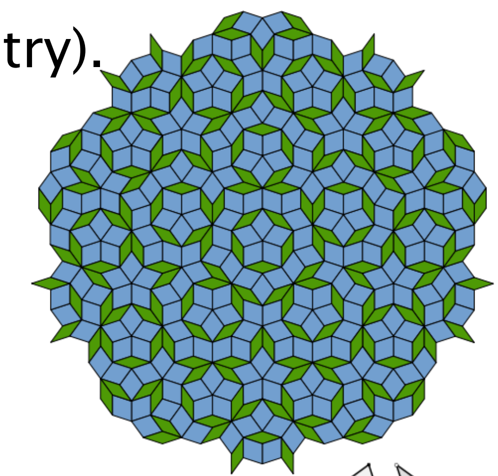


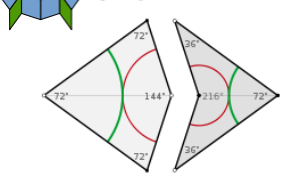
- Minimum spanning tree can be computed in $O(n^2)$ and is a lower bound on the shortest (open) tour

„Holistic” constraints: aperiodic tiling

- A tessellation of the plane or of any other space is a cover of the space by closed shapes, called tiles, that have disjoint interiors.

- A Penrose tiling:
  - It is non-periodic (lacks any translational symmetry).
  - It is self-similar.
  - It is a quasicrystal (as a physical structure).





- How can we find such exotic „patterns”?
- R.Penrose: Emperor’s new mind

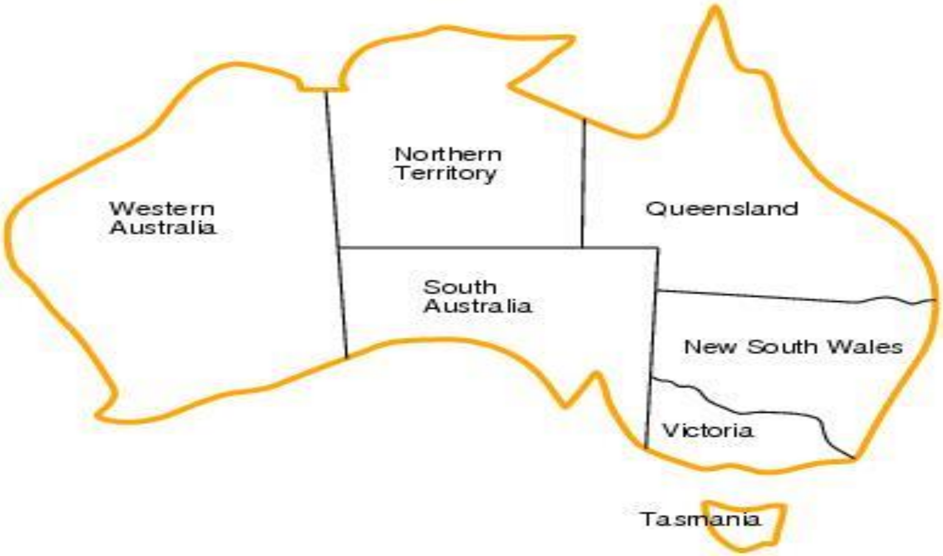
Constraint satisfaction problems

- What is a CSP?
  - Finite set of variables $V_1, V_2, \ldots, V_n$
  - Finite set of constraints $C_1, C_2, \ldots, C_m$
  - Nonempty domain of possible values for each variable $D_{V1}, D_{V2}, \ldots D_{Vn}$
  - Each constraint C_i limits the values that variables can take, e.g., $V_1 \neq V_2$
- A *state* is defined as an *assignment* of values to some or all variables.
- *Consistent assignment*: assignment does not not violate the constraints.

Constraint satisfaction problems

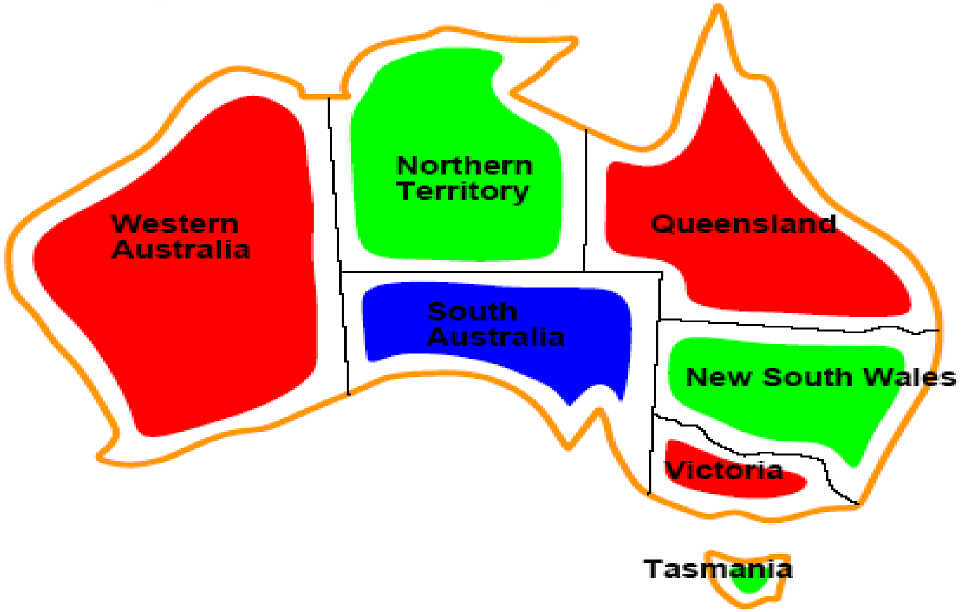
- An assignment is *complete* when every variable is mentioned.
- A *solution* to a CSP is a complete assignment that satisfies all constraints.
- Some CSPs require a solution that maximizes an *objective function*.
- Applications: Scheduling the time of observations on the Hubble Space Telescope, Floor planning, Map coloring, Cryptography

CSP example: map coloring



- Variables: *WA, NT, Q, NSW, V, SA, T*
- Domains: D_i*={red,green,blue}*
- Constraints:adjacent regions must have different colors.
  - E.g. $WA \neq NT$ (if the language allows this)
  - E.g. $(WA,NT) \neq$ *{(red,green),(red,blue),(green,red),…}*

CSP example: map coloring

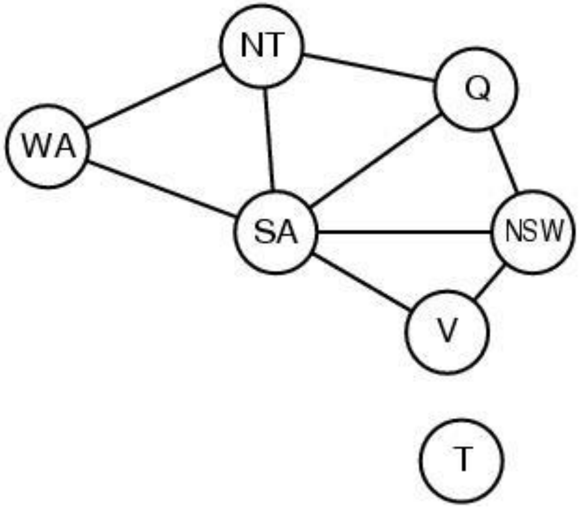


- Solutions are assignments satisfying all constraints, e.g.

  {WA=red,NT=green,Q=red,NSW=green,V=red,SA=blue, T=green}

Constraint graph

- CSP benefits
  - Standard representation pattern
  - Generic goal and successor functions
  - Generic heuristics (no domain specific expertise).



- Constraint graph = nodes are variables, edges show constraints.
  - **Graph can be used to simplify search.**
    - e.g. Tasmania is an independent subproblem.

Varieties of CSPs

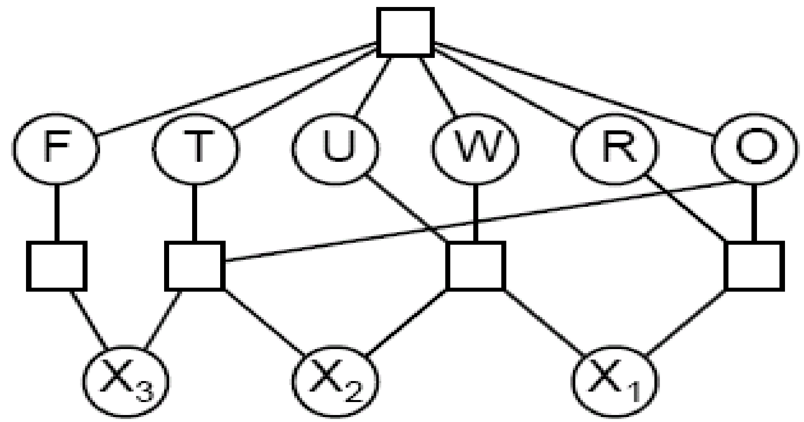
- Discrete variables
  - Finite domains; size $d \Rightarrow O(d^n)$ complete assignments.
    - E.g. Boolean CSPs, include. Boolean satisfiability (NP-complete).
  - Infinite domains (integers, strings, etc.)
    - E.g. job scheduling, variables are start/end days for each job
    - Need a constraint language e.g $StartJob_1 + 5 \leq StartJob_3$.
    - Linear constraints solvable, nonlinear undecidable.
- Continuous variables
  - e.g. start/end times for Hubble Telescope observations.
  - Linear constraints solvable in poly time by LP methods.

Varieties of constraints

- Unary constraints involve a single variable.
  - e.g. $SA \neq green$
- Binary constraints involve pairs of variables.
  - e.g. $SA \neq WA$
- Higher-order constraints involve 3 or more variables.
  - e.g. cryptharithmetic column constraints.
- Preference (soft constraints) e.g. *red* is better than *green* often representable by a cost for each variable assignment → constrained optimization problems.

Example; cryptharithmetic

```
    T W O
+   T W O
---------
  F O U R
```



Variables: $F\ T\ U\ W\ R\ O\ X_1\ X_2\ X_3$

Domains: $\{0,1,2,3,4,5,6,7,8,9\}$

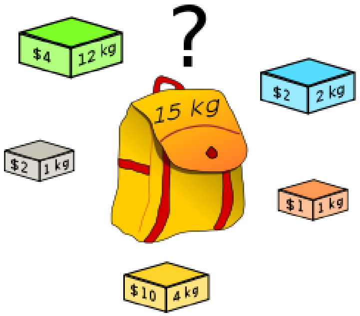
Constraints

alldiff(F,T,U,W,R,O)

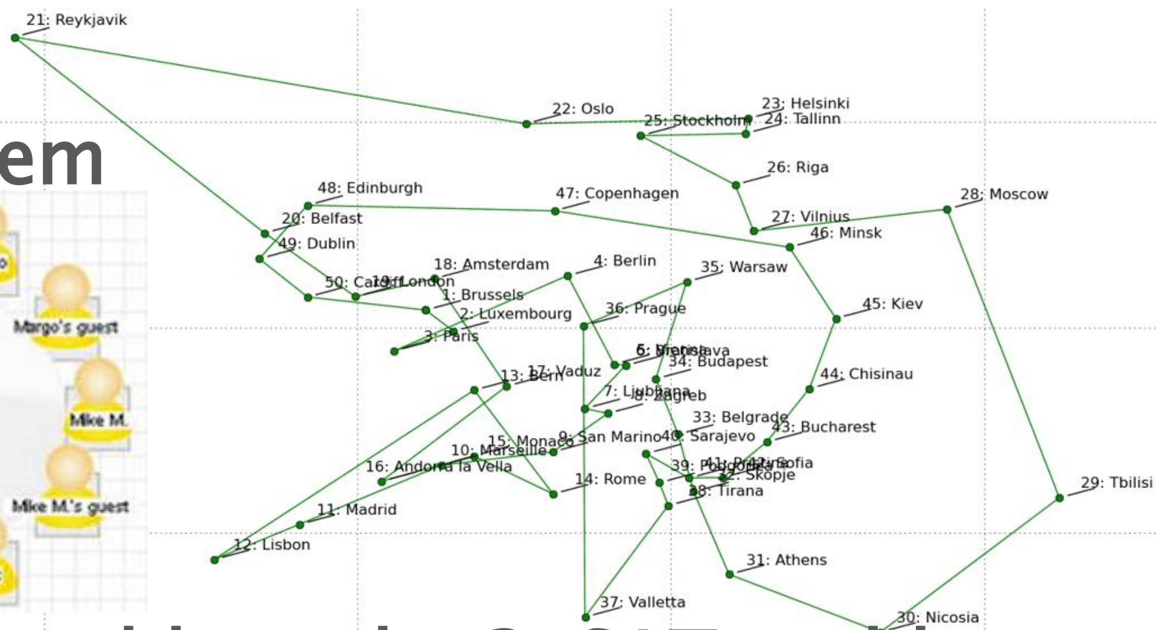
$O+O=R+10\cdot X_1$, etc.

CSP as combinatorial (optimization) problems

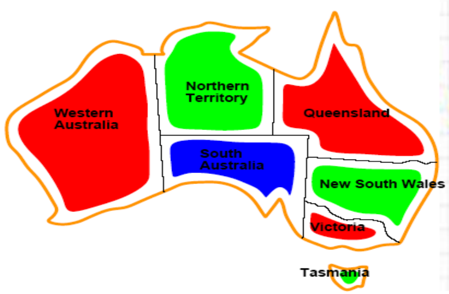
- **The „knapsack"/backpack problem**



- **The travelling sales man problem**



- The ménage problem



- The map coloring problem, the 3-SAT problem,...

$$(\neg B_{1,1} \lor P_{1,2} \lor P_{2,1}) \land (\neg P_{1,2} \lor B_{1,1}) \land (\neg P_{2,1} \lor B_{1,1})$$

CSP as a standard search problem

- A CSP can easily expressed as a standard search problem.
- Incremental formulation
  - *Initial State*: the empty assignment {}.
  - *Successor function*: Assign value to unassigned variable provided that there is not conflict.
  - *Goal test*: the current assignment is complete.
  - *Path cost*: as constant cost for every step.

CSP as a standard search problem

- This is the same for all CSP's !!!
- Solution is found at depth n (if there are n variables).
  - Hence depth first search can be used.
- Path is irrelevant, so optimization with complete state representation can also be used.
- Branching factor b at the top level is nd.
- $b=(n-l)d$ at depth l, hence $n!d^n$ leaves (only d^n complete assignments, $O(n^n)$, Stirling's approx.).

Commutativity

- CSPs are commutative.
  - The order of any given set of actions has no effect on the outcome.
  - Example: choose colors for Australian territories one at a time
    - [WA=red then NT=green] same as [NT=green then WA=red]
    - All CSP search algorithms consider a single variable assignment at a time $\Rightarrow$ there are d^n leaves.

Backtracking search

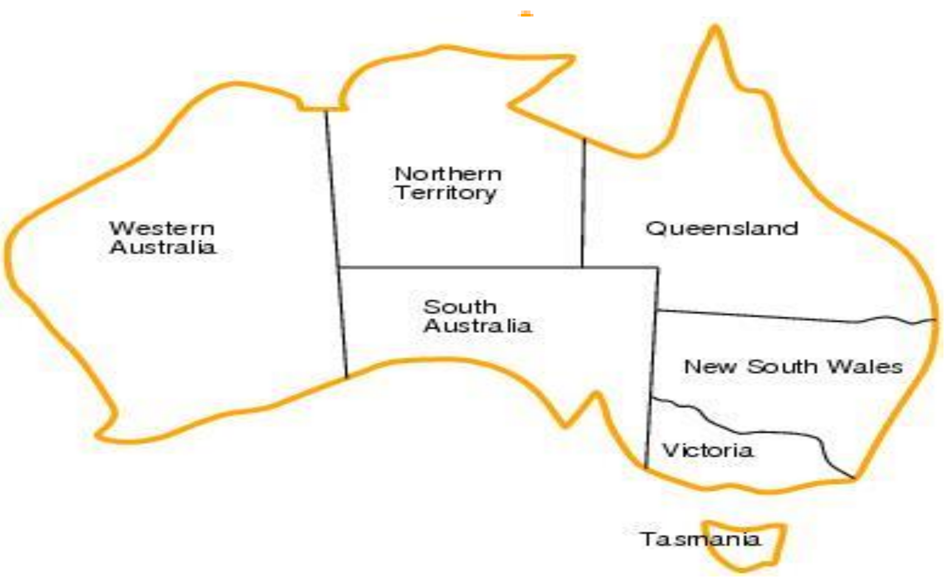
- Cfr. Depth-first search
- Chooses values for one variable at a time and backtracks when a variable has no legal values left to assign.
- Uninformed algorithm
  - No good general performance (see table p. 143)

Backtracking search

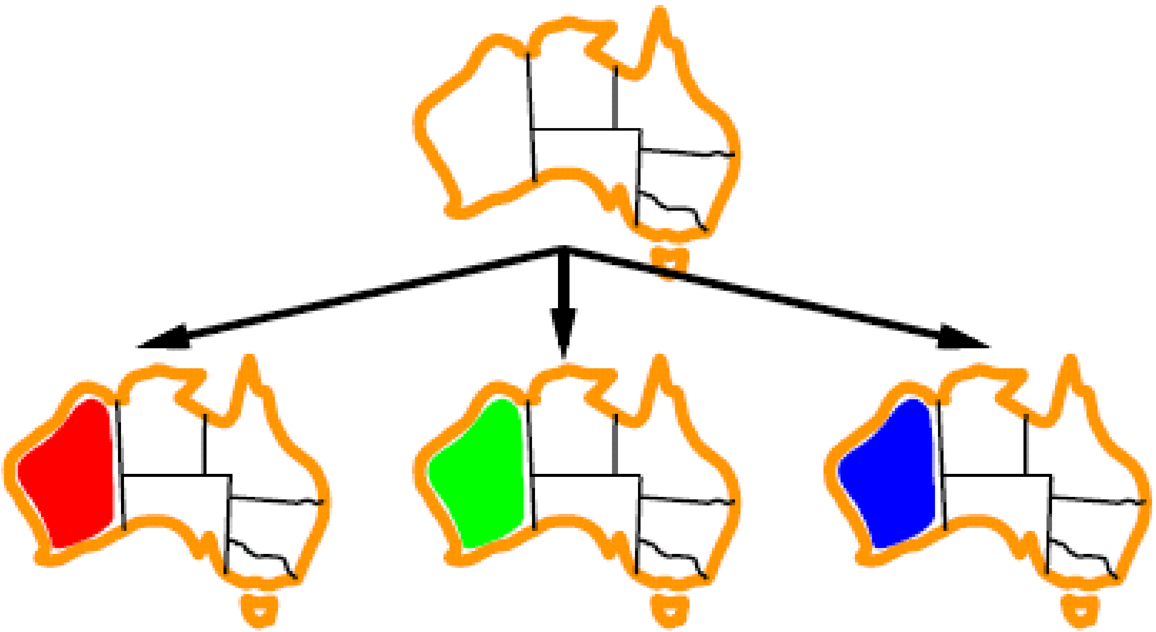
function BACKTRACKING-SEARCH(*csp*) **return** a solution or failure
 return RECURSIVE-BACKTRACKING(*{}* , *csp*)

function RECURSIVE-BACKTRACKING(*assignment, csp*) **return** a solution or failure
 if *assignment* is complete **then return** *assignment*
 var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[*csp*],*assignment*,*csp*)
 for each *value* **in** ORDER-DOMAIN-VALUES(*var, assignment, csp*) **do**
  if *value* is consistent with *assignment* according to CONSTRAINTS[*csp*] **then**
   add *{var=value}* to assignment
   result ← RRECURSIVE-BACTRACKING(*assignment, csp*)
   if *result* ≠ *failure* **then return** *result*
   remove *{var=value}* from *assignment*
 return *failure*

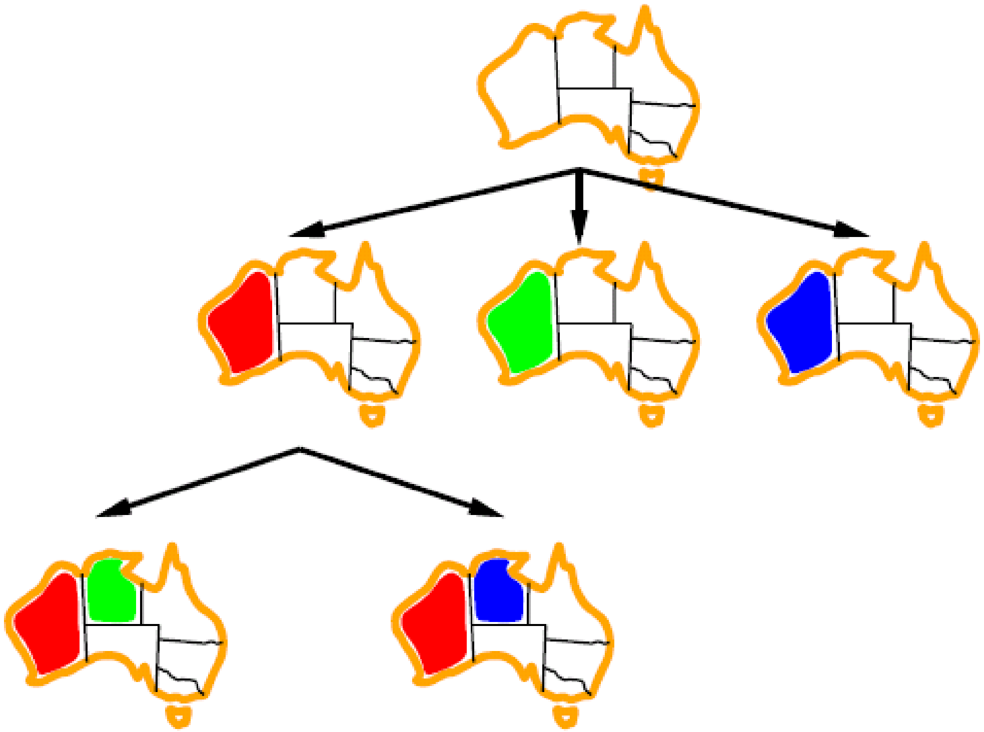
Backtracking example



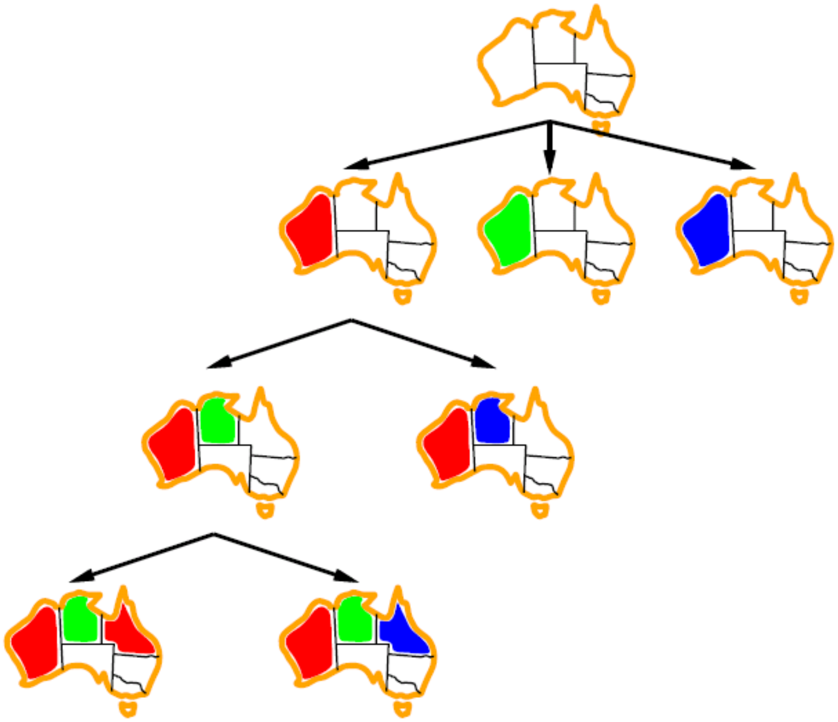
Backtracking example



Backtracking example



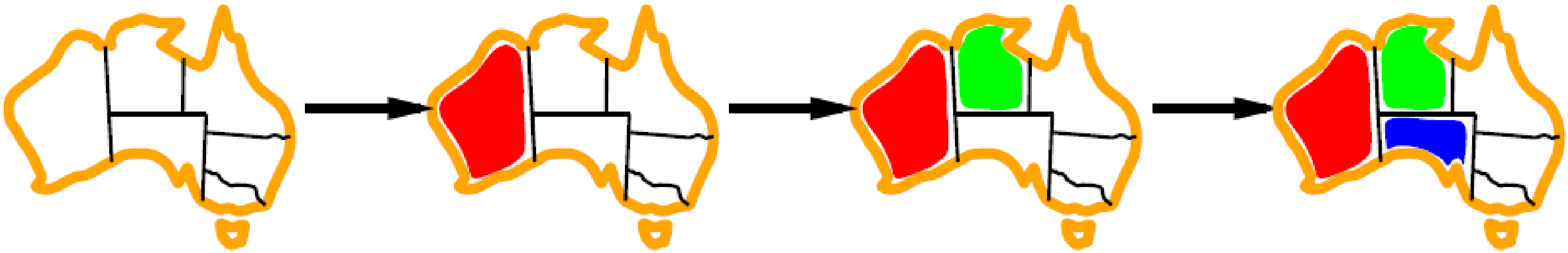
Backtracking example



Improving backtracking efficiency

- Previous improvements → introduce heuristics
- General-purpose methods can give huge gains in speed:
  - Which variable should be assigned next?
  - In what order should its values be tried?
  - Can we detect inevitable failure early?
  - Can we take advantage of problem structure?

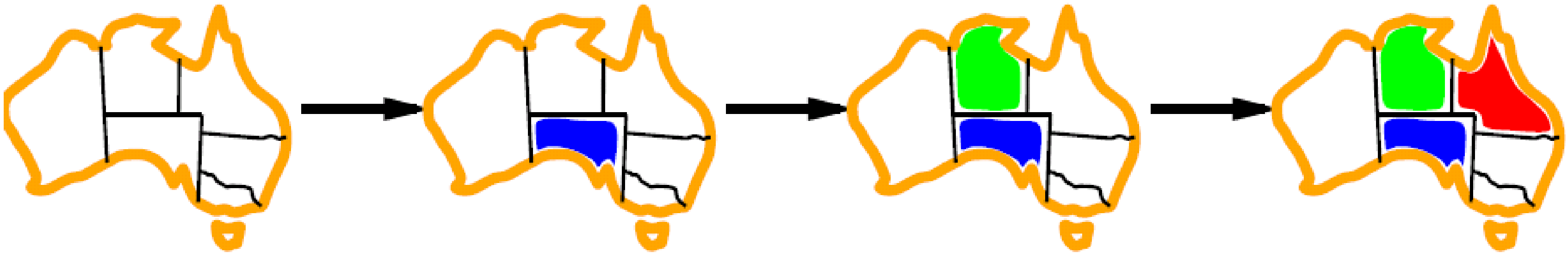
Most constraining variable (Minimum remaining values)



var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[*csp*],*assignment*,*csp*)

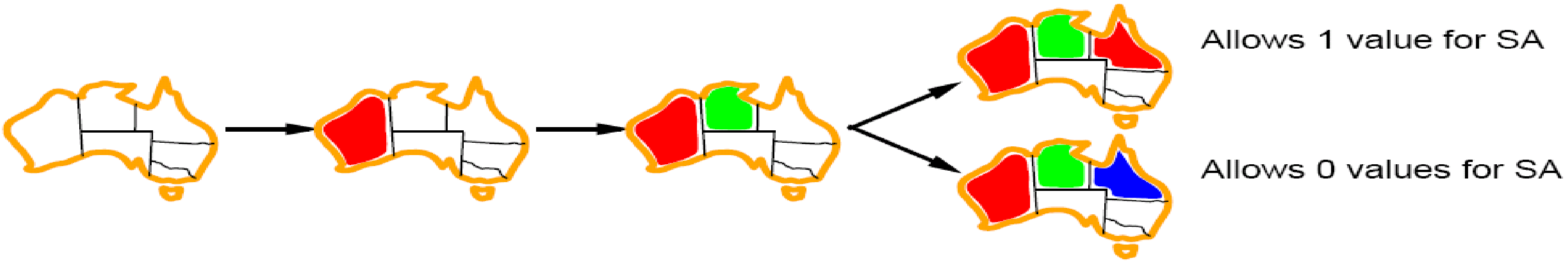
- A.k.a. most constrained variable heuristic („fail fast")
- *Rule*: choose variable with the fewest legal moves
- *Which variable shall we try first?*

Degree heuristic



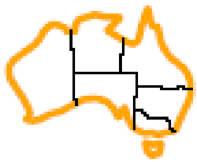
- Use degree heuristic
- *Rule*: select variable that is involved in the largest number of constraints on other unassigned variables.
- Degree heuristic is very useful as a tie breaker.
- *In what order should its values be tried?*

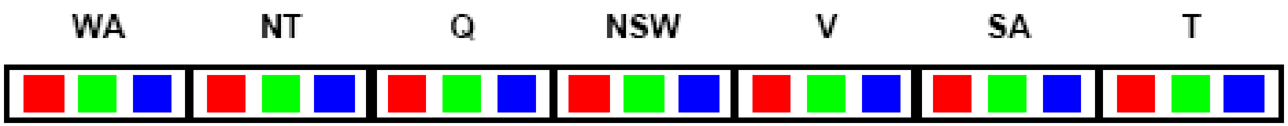
Least constraining value



- Least constraining value heuristic
- Rule: given a variable choose the least constraing value i.e. the one that leaves the maximum flexibility for subsequent variable assignments.

Forward checking





- Can we detect inevitable failure early?
  - *And avoid it later?*
- *Forward checking idea:* keep track of remaining legal values for unassigned variables.
- Terminate search when any variable has no legal values.

K-consistency

- A CSP is k-consistent if for any set of k-1 variables and for any consistent assignment to those variables, a consistent value can always be assigned to any kth variable.
- A graph is strongly k-consistent if
  - It is k-consistent and
  - Is also (k-1) consistent, (k-2) consistent, … all the way down to 1-consistent.

- YET *no free lunch*: any algorithm for establishing n-consistency must take time exponential in n, in the worst case.

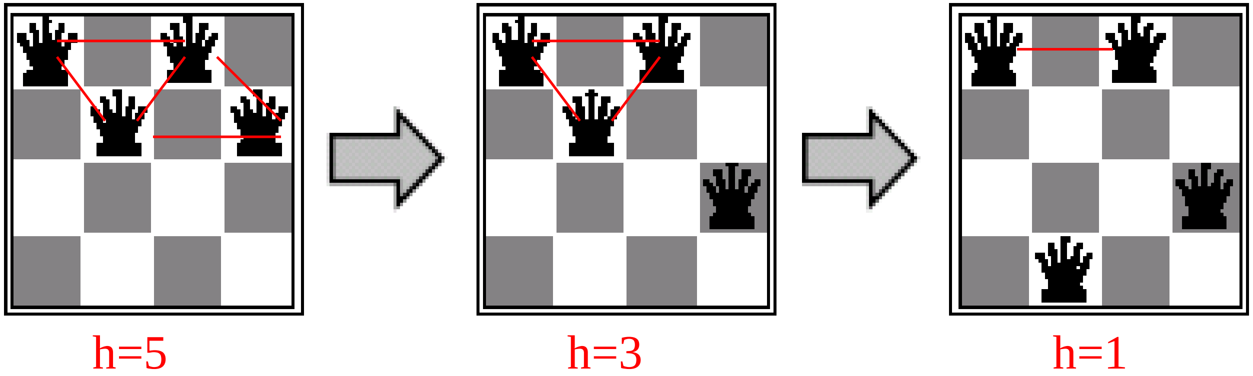
Local search (optimization) for CSP

- Use complete-state representation
- For CSPs
  - allow states with unsatisfied constraints
  - operators **reassign** variable values
- Variable selection: randomly select any conflicted variable
- Value selection: *min-conflicts heuristic*
  - Select new value that results in a minimum number of conflicts with the other variables

Local search for CSP

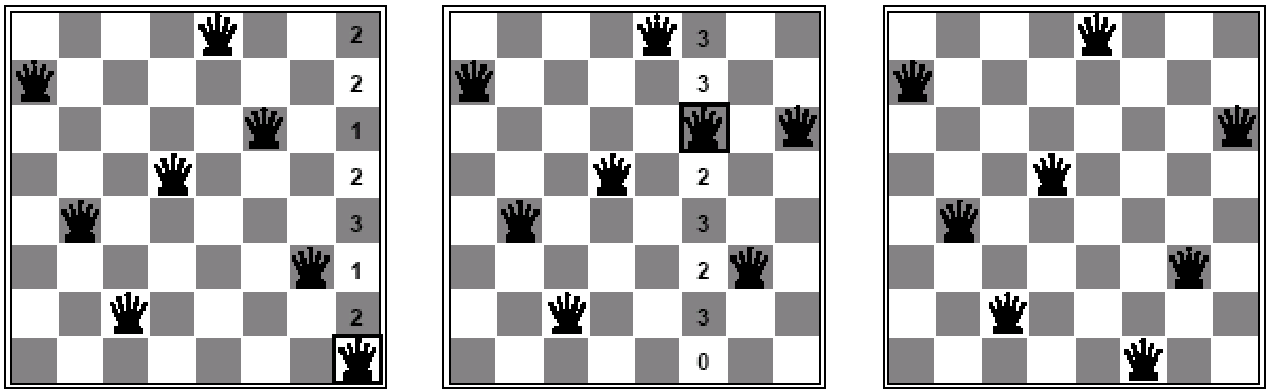
```
function MIN-CONFLICTS(csp, max_steps) return solution or failure
   inputs: csp, a constraint satisfaction problem
           max_steps, the number of steps allowed before giving up

   current ← an initial complete assignment for csp
   for i = 1 to max_steps do
           if current is a solution for csp then return current
           var ← a randomly chosen, conflicted variable from VARIABLES[csp]
           value ← the value v for var that minimizes CONFLICTS(var,v,current,csp)
           set var = value in current
   return faiilure
```

Min-conflicts example 1



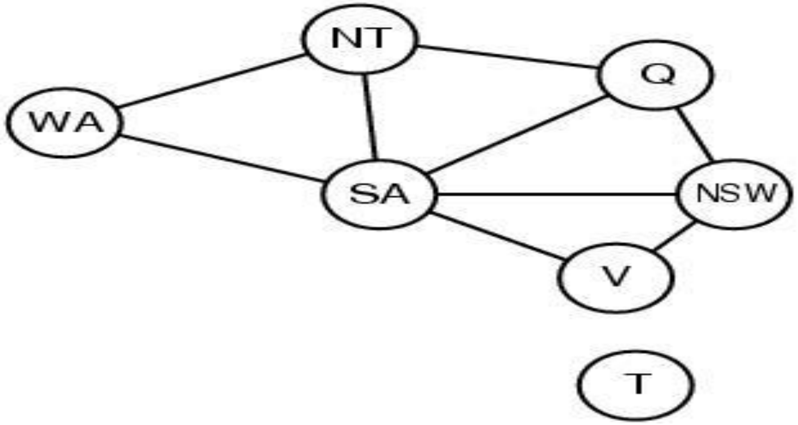
- Use of min-conflicts heuristic in hill-climbing.

Min-conflicts example 2



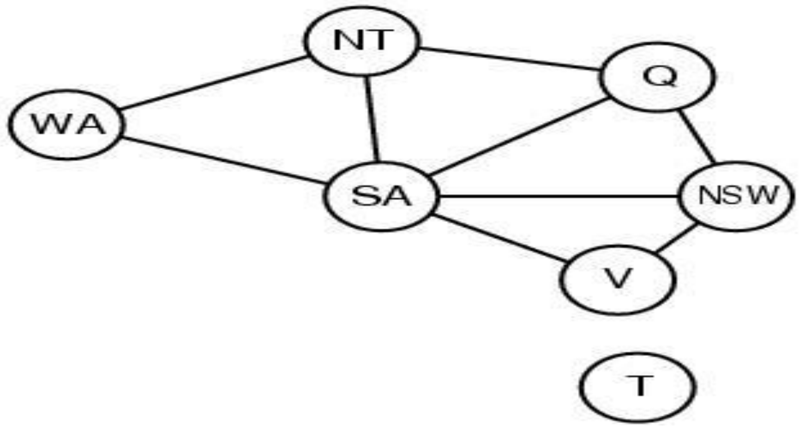
- A two-step solution for an 8-queens problem using min-conflicts heuristic.
- At each stage a queen is chosen for reassignment in its column.
- The algorithm moves the queen to the min-conflict square breaking ties randomly.

Problem structure



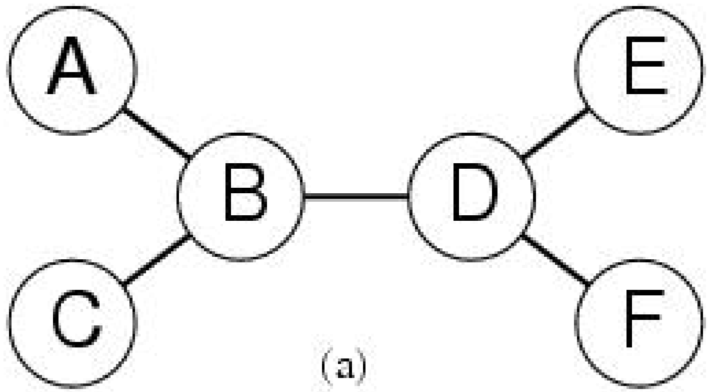
- *How can the problem structure help to find a solution quickly?*
- Subproblem identification is important:
  - Coloring Tasmania and mainland are independent subproblems
  - Identifiable as connected components of constrained graph.
- Improves performance

Problem structure



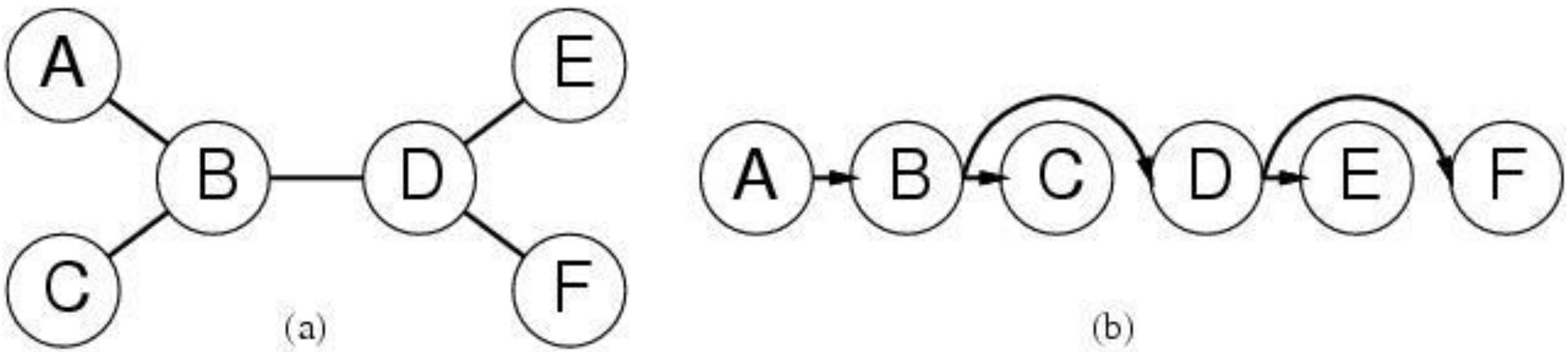
- Suppose each problem has *c* variables out of a total of *n*.
- Worst case solution cost is $O(n/c\ d^c)$, i.e. linear in *n*
  - Instead of $O(d^n)$, exponential in *n*
- E.g. $n= 80, c= 20, d=2$
  - 2^{80} = 4 billion years at 1 million nodes/sec.
  - $4 * 2^{20}$= .4 second at 1 million nodes/sec

Tree-structured CSPs



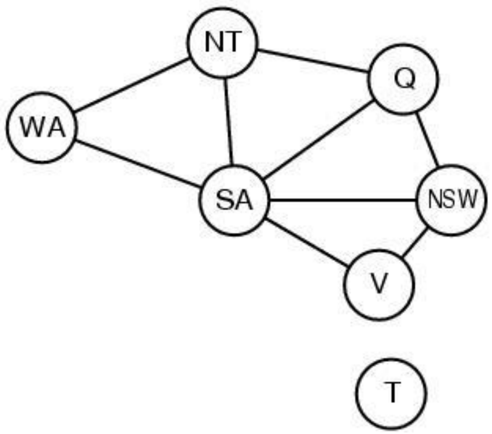
- Theorem: if the constraint graph has no loops then CSP can be solved in $O(nd^2)$ time
- Compare difference with general CSP, where worst case is $O(d^n)$

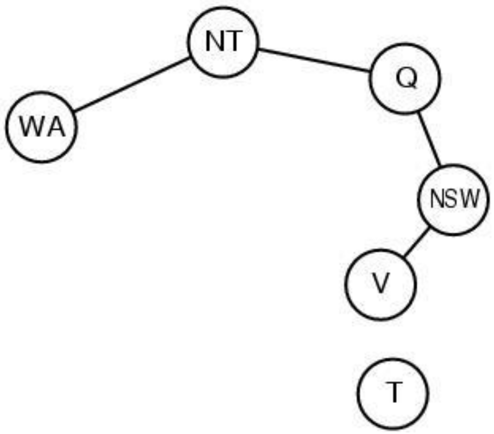
Tree-structured CSPs



- In most cases subproblems of a CSP are connected as a tree
- Any tree-structured CSP can be solved in time linear in the number of variables.
  - Choose a variable as root, order variables from root to leaves such that every node's parent precedes it in the ordering.
  - For j from n down to 2, apply REMOVE-INCONSISTENT-VALUES(Parent(X_j),X_j)
  - For j from 1 to n assign X_j consistently with Parent(X_j)

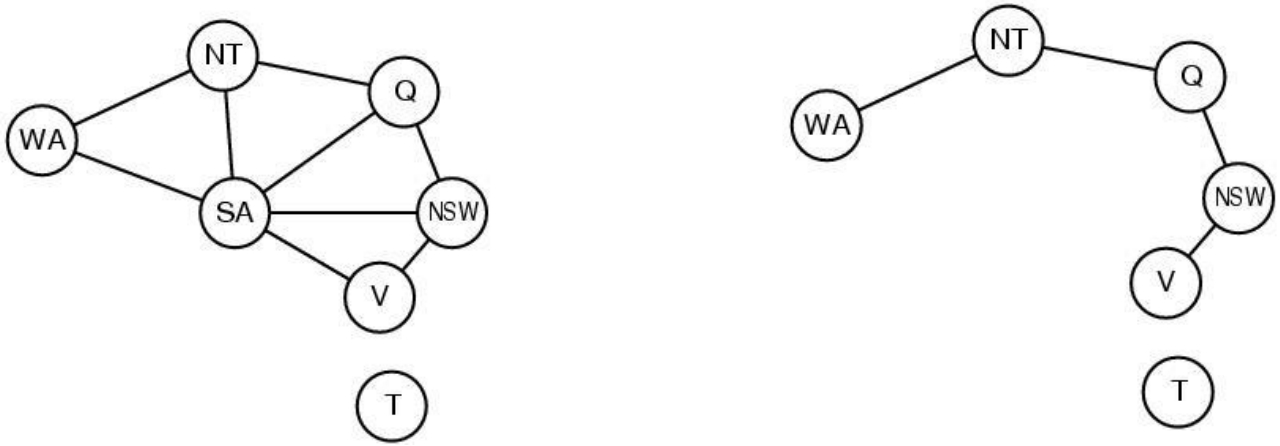
Nearly tree-structured CSPs





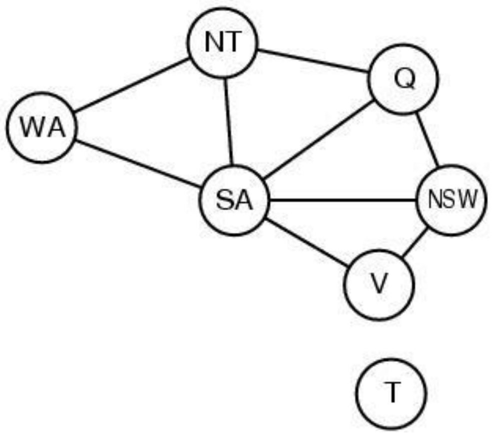
- *Can more general constraint graphs be reduced to trees?*
- Two approaches:
  - Remove certain nodes
  - Collapse certain nodes

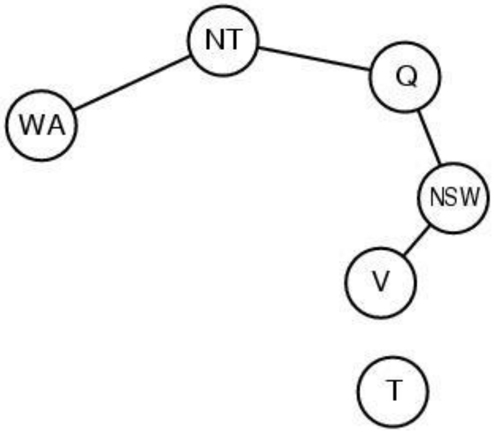
Nearly tree-structured CSPs



- Idea: assign values to some variables so that the remaining variables form a tree.
- Assume that we assign *{SA=x}* ← *cycle cutset*
  - And remove any values from the other variables that are inconsistent.
  - The selected value for SA could be the wrong one so we have to try all of them

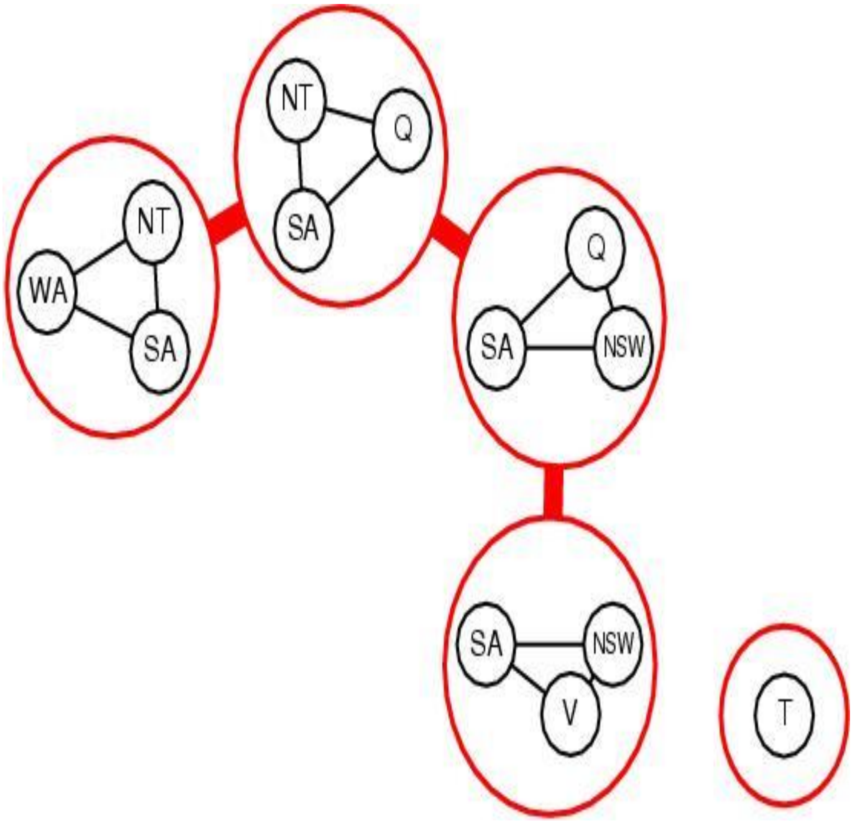
Nearly tree-structured CSPs





- This approach is worthwhile if cycle cutset is small.
- Finding the smallest cycle cutset is NP-hard
  - Approximation algorithms exist
- This approach is called *cutset conditioning*.

Nearly tree-structured CSPs



- Tree decomposition of the constraint graph in a set of connected subproblems.
- Each subproblem is solved independently
- Resulting solutions are combined.
- Necessary requirements:
  - Every variable appears in at least one of the subproblems.
  - If two variables are connected in the original problem, they must appear together in at least one subproblem.
  - If a variable appears in two subproblems, it must appear in each node on the path.

Summary

- CSPs are a special kind of problem: states defined by values of a fixed set of variables, goal test defined by constraints on variable values
- Backtracking=depth-first search with one variable assigned per node
- Variable ordering and value selection heuristics help significantly
- Forward checking prevents assignments that lead to failure.
- Constraint propagation does additional work to constrain values and detect inconsistencies.
- The CSP representation allows analysis of problem structure.
- Tree structured CSPs can be solved in linear time.
- Iterative min-conflicts is usually effective in practice.