

EXACT INFERENCE IN GENERAL BAYESIAN NETWORKS, IN NAIVE BNS AND IN HIDDEN MARKOV MODELS

AI: EXACT INFERENCE IN BNS

Outline

- ◇ Types of inference in (causal) BNs
- ◇ Exact inference by enumeration
- ◇ Hardness of exact inference in general BNs
- ◇ Linear time diagnostic inference in Naive BNs
- ◇ Types of (observational) inference in Hidden Markov Models
- ◇ Approximate inference by stochastic simulation
- ◇ Approximate inference by Markov chain Monte Carlo

Inference tasks

Simple queries: compute posterior marginal $\mathbf{P}(X_i|\mathbf{E} = \mathbf{e})$

e.g., $P(\text{NoGas}|\text{Gauge} = \text{empty}, \text{Lights} = \text{on}, \text{Starts} = \text{false})$

Conjunctive queries: $\mathbf{P}(X_i, X_j|\mathbf{E} = \mathbf{e}) = \mathbf{P}(X_i|\mathbf{E} = \mathbf{e})\mathbf{P}(X_j|X_i, \mathbf{E} = \mathbf{e})$

Optimal decisions: decision networks include utility information;
probabilistic inference required for $P(\text{outcome}|\text{action}, \text{evidence})$

Value of information: which evidence to seek next?

Sensitivity analysis: which probability values are most critical?

Explanation: why do I need a new starter motor?

Causal inference: what is the effect of an intervention?

Counterfactual inference: what would have been the effect of a hypothetical/imagery past intervention&observation?

Inference by enumeration: principle

Let \mathbf{X} be all the variables. Typically, we want the posterior joint distribution of the query variables \mathbf{Y} given specific values \mathbf{e} for the evidence variables \mathbf{E} .

Let the hidden variables be $\mathbf{H} = \mathbf{X} - \mathbf{Y} - \mathbf{E}$.

Then the required summation of joint entries is done by summing out the hidden variables:

$$\mathbf{P}(\mathbf{Y}|\mathbf{E} = \mathbf{e}) = \alpha \mathbf{P}(\mathbf{Y}, \mathbf{E} = \mathbf{e}) = \alpha \sum_{\mathbf{h}} \mathbf{P}(\mathbf{Y}, \mathbf{E} = \mathbf{e}, \mathbf{H} = \mathbf{h})$$

The terms in the summation are joint entries!

Obvious problems:

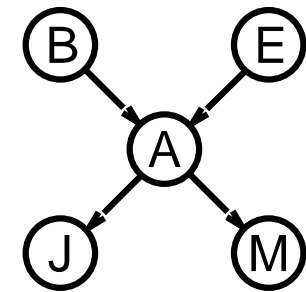
- 1) Worst-case time complexity $O(d^n)$ where d is the largest arity
- 2) Space complexity $O(d^n)$ to store the joint distribution
- 3) How to find the numbers for $O(d^n)$ entries???

Inference by enumeration: goal oriented

Slightly intelligent way to sum out variables from the joint without actually constructing its explicit representation

Simple query on the burglary network:

$$\begin{aligned} & \mathbf{P}(B|j, m) \\ &= \mathbf{P}(B, j, m) / P(j, m) \\ &= \alpha \mathbf{P}(B, j, m) \\ &= \alpha \sum_e \sum_a \mathbf{P}(B, e, a, j, m) \end{aligned}$$



Rewrite full joint entries using product of CPT entries:

$$\begin{aligned} & \mathbf{P}(B|j, m) \\ &= \alpha \sum_e \sum_a \mathbf{P}(B)P(e)\mathbf{P}(a|B, e)P(j|a)P(m|a) \\ &= \alpha \mathbf{P}(B) \sum_e P(e) \sum_a \mathbf{P}(a|B, e)P(j|a)P(m|a) \end{aligned}$$

Recursive depth-first enumeration: $O(n)$ space, $O(d^n)$ time

Complexity of exact inference

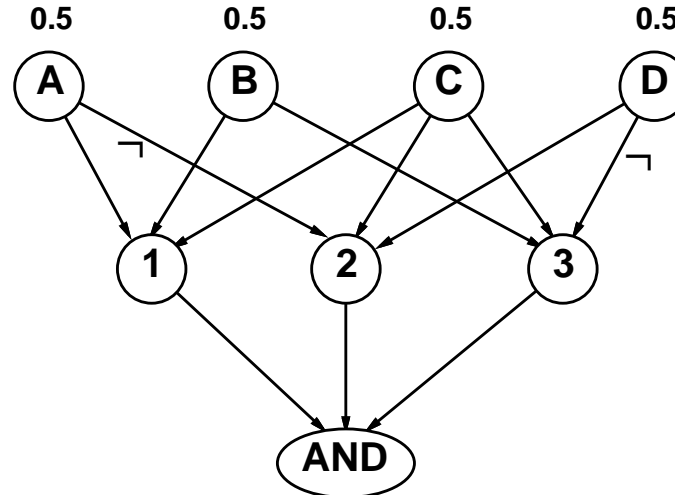
Singly connected networks (or **polytrees**):

- any two nodes are connected by at most one (undirected) path
- time and space cost of exact inference $O(d^k n)$

Multiply connected networks:

- can reduce 3SAT to exact inference \Rightarrow NP-hard
- equivalent to **counting** 3SAT models \Rightarrow #P-complete

1. $A \vee B \vee C$
2. $C \vee D \vee \neg A$
3. $B \vee C \vee \neg D$



Diagnostic inference in Naive BNs

Useful for assessing **diagnostic** probability from **causal** probabilities:

$$P(\textit{Cause}|\textit{Effect}_{1:n})$$

$$= \frac{P(\textit{Cause}) \prod_{i=1}^n P(\textit{Effect}_i|\textit{Cause})}{P(\textit{Effect}_{1:n})}$$

$$\propto P(\textit{Cause}) \prod_{i=1}^n P(\textit{Effect}_i|\textit{Cause})$$

Hidden Markov Models

The world changes; we need to track and predict it

Diabetes management vs vehicle diagnosis

Basic idea: copy state and evidence variables for each time step

\mathbf{X}_t = set of unobservable state variables at time t
e.g., *BloodSugar_t*, *StomachContents_t*, etc.

\mathbf{E}_t = set of observable evidence variables at time t
e.g., *MeasuredBloodSugar_t*, *PulseRate_t*, *FoodEaten_t*

This assumes **discrete time**; step size depends on problem

Notation: $\mathbf{X}_{a:b} = \mathbf{X}_a, \mathbf{X}_{a+1}, \dots, \mathbf{X}_{b-1}, \mathbf{X}_b$

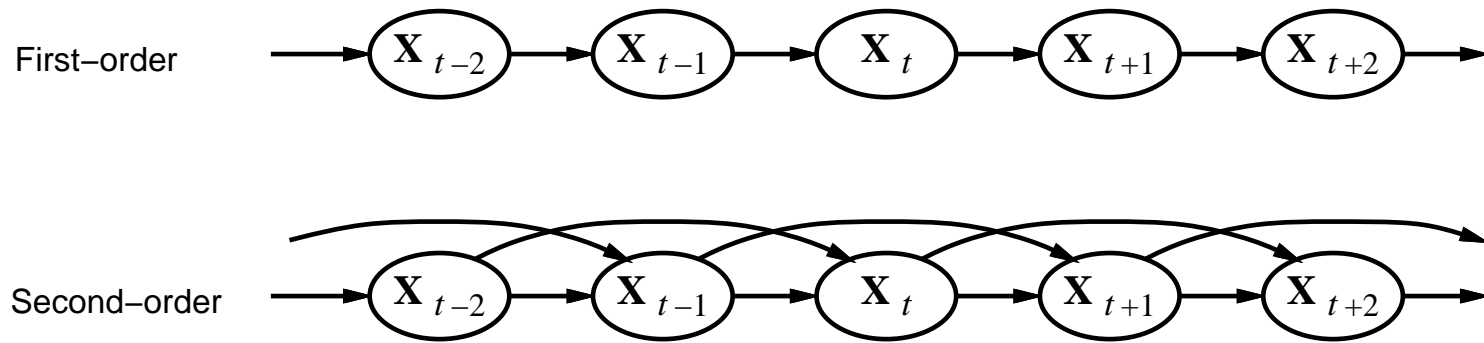
Markov processes (Markov chains)

Construct a Bayes net from these variables: parents?

Markov assumption: \mathbf{X}_t depends on **bounded** subset of $\mathbf{X}_{0:t-1}$

First-order Markov process: $\mathbf{P}(\mathbf{X}_t | \mathbf{X}_{0:t-1}) = \mathbf{P}(\mathbf{X}_t | \mathbf{X}_{t-1})$

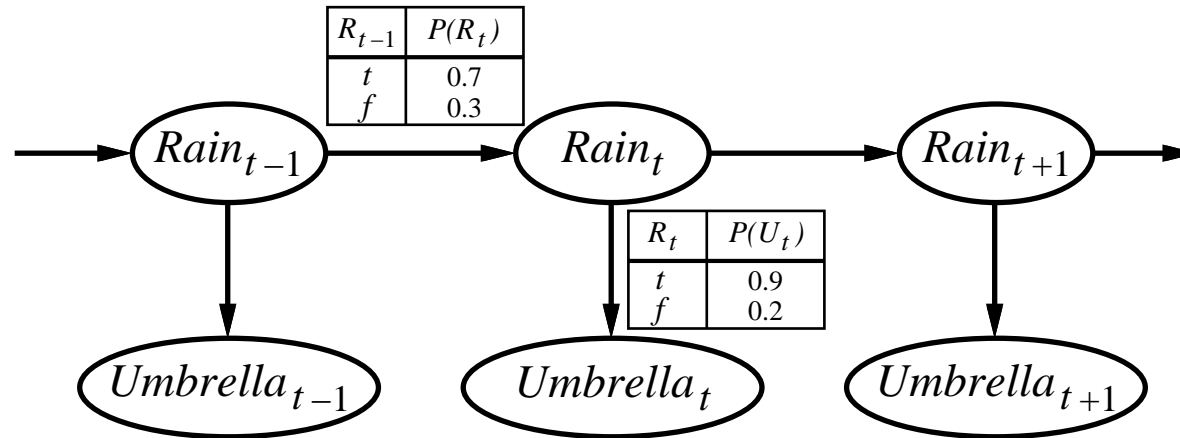
Second-order Markov process: $\mathbf{P}(\mathbf{X}_t | \mathbf{X}_{0:t-1}) = \mathbf{P}(\mathbf{X}_t | \mathbf{X}_{t-2}, \mathbf{X}_{t-1})$



Sensor Markov assumption: $\mathbf{P}(\mathbf{E}_t | \mathbf{X}_{0:t}, \mathbf{E}_{0:t-1}) = \mathbf{P}(\mathbf{E}_t | \mathbf{X}_t)$

Stationary process: transition model $\mathbf{P}(\mathbf{X}_t | \mathbf{X}_{t-1})$ and sensor model $\mathbf{P}(\mathbf{E}_t | \mathbf{X}_t)$ fixed for all t

Example



First-order Markov assumption not exactly true in real world!

Possible fixes:

1. **Increase order** of Markov process
2. **Augment state**, e.g., add $Temp_t$, $Pressure_t$

Example: robot motion.

Augment position and velocity with $Battery_t$

Inference tasks

Filtering: $\mathbf{P}(\mathbf{X}_t | \mathbf{e}_{1:t})$

belief state—input to the decision process of a rational agent

Prediction: $\mathbf{P}(\mathbf{X}_{t+k} | \mathbf{e}_{1:t})$ for $k > 0$

evaluation of possible action sequences;
like filtering without the evidence

Smoothing: $\mathbf{P}(\mathbf{X}_k | \mathbf{e}_{1:t})$ for $0 \leq k < t$

better estimate of past states, essential for learning

Most likely explanation: $\arg \max_{\mathbf{x}_{1:t}} P(\mathbf{x}_{1:t} | \mathbf{e}_{1:t})$

speech recognition, decoding with a noisy channel

Filtering

Aim: devise a **recursive** state estimation algorithm:

$$\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t+1}) = f(\mathbf{e}_{t+1}, \mathbf{P}(\mathbf{X}_t|\mathbf{e}_{1:t}))$$

$$\begin{aligned}\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t+1}) &= \mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t}, \mathbf{e}_{t+1}) \\ &= \alpha \mathbf{P}(\mathbf{e}_{t+1}|\mathbf{X}_{t+1}, \mathbf{e}_{1:t}) \mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t}) \\ &= \alpha \mathbf{P}(\mathbf{e}_{t+1}|\mathbf{X}_{t+1}) \mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t})\end{aligned}$$

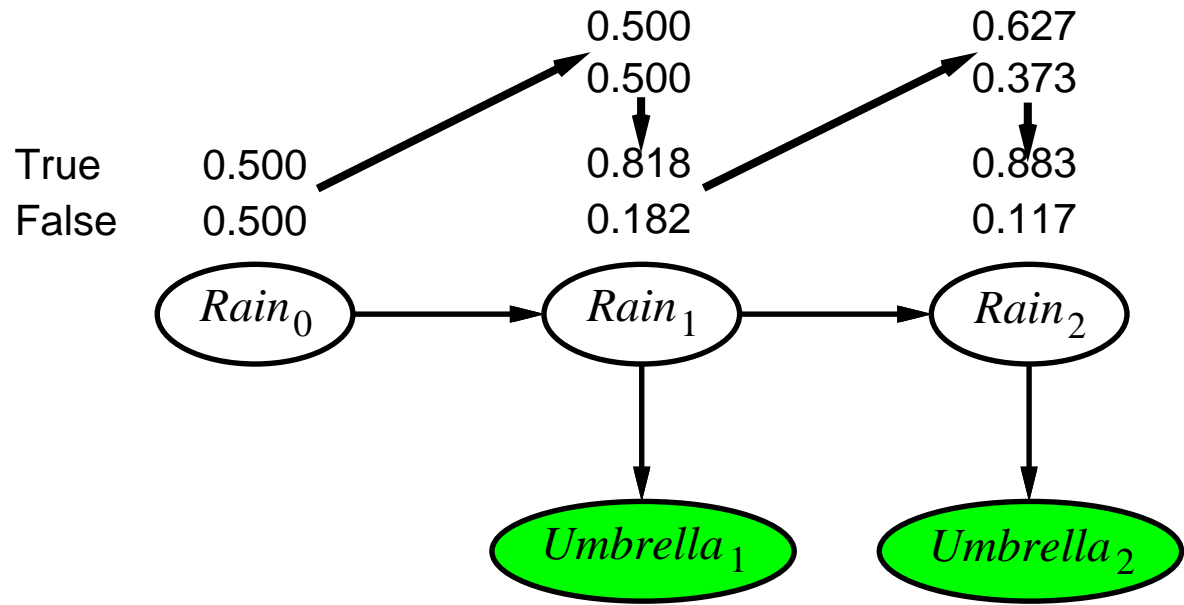
I.e., **prediction** + **estimation**. Prediction by summing out \mathbf{X}_t :

$$\begin{aligned}\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t+1}) &= \alpha \mathbf{P}(\mathbf{e}_{t+1}|\mathbf{X}_{t+1}) \sum_{\mathbf{x}_t} \mathbf{P}(\mathbf{X}_{t+1}|\mathbf{x}_t, \mathbf{e}_{1:t}) P(\mathbf{x}_t|\mathbf{e}_{1:t}) \\ &= \alpha \mathbf{P}(\mathbf{e}_{t+1}|\mathbf{X}_{t+1}) \sum_{\mathbf{x}_t} \mathbf{P}(\mathbf{X}_{t+1}|\mathbf{x}_t) P(\mathbf{x}_t|\mathbf{e}_{1:t})\end{aligned}$$

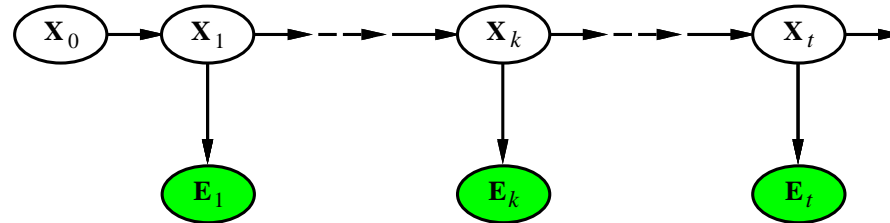
$\mathbf{f}_{1:t+1} = \text{FORWARD}(\mathbf{f}_{1:t}, \mathbf{e}_{t+1})$ where $\mathbf{f}_{1:t} = \mathbf{P}(\mathbf{X}_t|\mathbf{e}_{1:t})$

Time and space **constant** (independent of t)

Filtering example



Smoothing



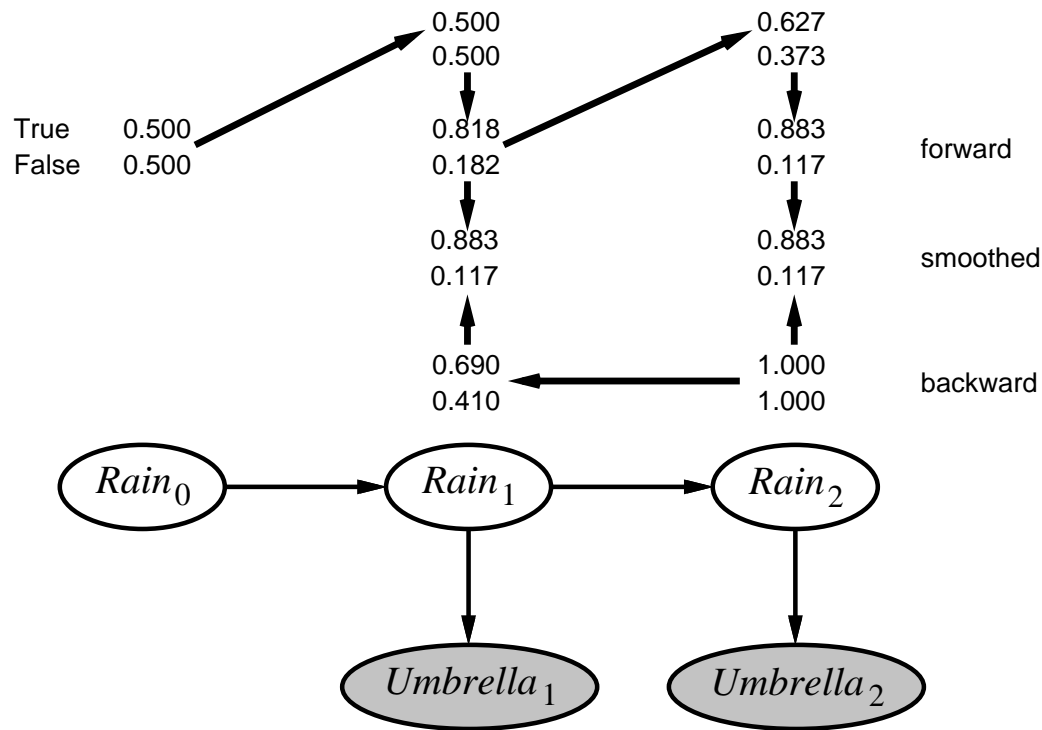
Divide evidence $\mathbf{e}_{1:t}$ into $\mathbf{e}_{1:k}$, $\mathbf{e}_{k+1:t}$:

$$\begin{aligned}
 \mathbf{P}(\mathbf{X}_k | \mathbf{e}_{1:t}) &= \mathbf{P}(\mathbf{X}_k | \mathbf{e}_{1:k}, \mathbf{e}_{k+1:t}) \\
 &= \alpha \mathbf{P}(\mathbf{X}_k | \mathbf{e}_{1:k}) \mathbf{P}(\mathbf{e}_{k+1:t} | \mathbf{X}_k, \mathbf{e}_{1:k}) \\
 &= \alpha \mathbf{P}(\mathbf{X}_k | \mathbf{e}_{1:k}) \mathbf{P}(\mathbf{e}_{k+1:t} | \mathbf{X}_k) \\
 &= \alpha \mathbf{f}_{1:k} \mathbf{b}_{k+1:t}
 \end{aligned}$$

Backward message computed by a backwards recursion:

$$\begin{aligned}
 \mathbf{P}(\mathbf{e}_{k+1:t} | \mathbf{X}_k) &= \sum_{\mathbf{x}_{k+1}} \mathbf{P}(\mathbf{e}_{k+1:t} | \mathbf{X}_k, \mathbf{x}_{k+1}) \mathbf{P}(\mathbf{x}_{k+1} | \mathbf{X}_k) \\
 &= \sum_{\mathbf{x}_{k+1}} P(\mathbf{e}_{k+1:t} | \mathbf{x}_{k+1}) \mathbf{P}(\mathbf{x}_{k+1} | \mathbf{X}_k) \\
 &= \sum_{\mathbf{x}_{k+1}} P(\mathbf{e}_{k+1} | \mathbf{x}_{k+1}) P(\mathbf{e}_{k+2:t} | \mathbf{x}_{k+1}) \mathbf{P}(\mathbf{x}_{k+1} | \mathbf{X}_k)
 \end{aligned}$$

Smoothing example



Forward-backward algorithm: cache forward messages along the way
 Time linear in t (polytree inference), space $O(t|f|)$

Most likely explanation

Most likely sequence \neq sequence of most likely states!!!!

Most likely path to each \mathbf{x}_{t+1}

= most likely path to **some** \mathbf{x}_t plus one more step

$$\begin{aligned} & \max_{\mathbf{x}_1 \dots \mathbf{x}_t} \mathbf{P}(\mathbf{x}_1, \dots, \mathbf{x}_t, \mathbf{X}_{t+1} | \mathbf{e}_{1:t+1}) \\ & = \mathbf{P}(\mathbf{e}_{t+1} | \mathbf{X}_{t+1}) \max_{\mathbf{x}_t} \left(\mathbf{P}(\mathbf{X}_{t+1} | \mathbf{x}_t) \max_{\mathbf{x}_1 \dots \mathbf{x}_{t-1}} P(\mathbf{x}_1, \dots, \mathbf{x}_{t-1}, \mathbf{x}_t | \mathbf{e}_{1:t}) \right) \end{aligned}$$

Identical to filtering, except $\mathbf{f}_{1:t}$ replaced by

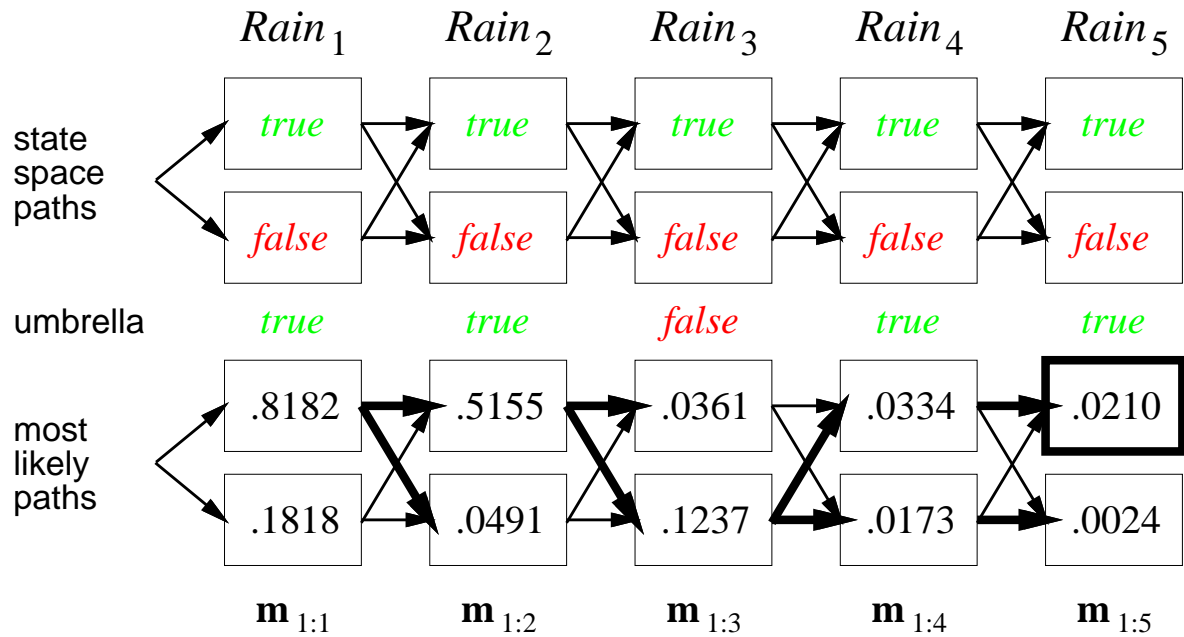
$$\mathbf{m}_{1:t} = \max_{\mathbf{x}_1 \dots \mathbf{x}_{t-1}} \mathbf{P}(\mathbf{x}_1, \dots, \mathbf{x}_{t-1}, \mathbf{X}_t | \mathbf{e}_{1:t}),$$

I.e., $\mathbf{m}_{1:t}(i)$ gives the probability of the most likely path to state i .

Update has sum replaced by max, giving the **Viterbi algorithm**:

$$\mathbf{m}_{1:t+1} = \mathbf{P}(\mathbf{e}_{t+1} | \mathbf{X}_{t+1}) \max_{\mathbf{x}_t} \left(\mathbf{P}(\mathbf{X}_{t+1} | \mathbf{x}_t) \mathbf{m}_{1:t} \right)$$

Viterbi example



Summary

Exact inference:

- polytime on polytrees (NBNs, HMMs)
- NP-hard on general graphs