The variance of the power density spectrum of periodic signals*

I. Kollár
Department of Measurement and Instrument Engineering, Technical University of Budapest, Budapest, Hungary

The measurement of the spectrum of periodic signals is generally done by assuming zero variance. However, since the phase is often random, this is not exactly true, though generally the variance is small indeed. In the paper, expressions of the variance of the periodogram-based spectral estimator are derived, in the case of different windows, for periodic and for mixed periodic-Gaussian signals. Some examples are given for the cases when the variance is significant.

1 Introduction

Well-known results concerning the variance of power density spectrum measurements of stochastic signals using the periodogram method can be found in the literature (Bendat and Piersol, 1971). The (not averaged) estimator is the so-called modified periodogram:

$$\hat{S}(f) = \frac{1}{T_e}\overline{X_w(f)}\,X_w(f), \qquad \ldots(1)$$

In this expression, T_e is the equivalent measurement time:

$$T_e = \int_{-\infty}^{\infty} w^2(t)\,dt = \int_{-\infty}^{\infty} W^2(f)\,df$$

and

$$X_w(f) = \int_{-\infty}^{\infty} x(t)\,w(t)\exp(-j2\pi ft)\,dt.$$

The variance of Expn (1) is:

$$\operatorname{var}\{\hat{S}_n(f)\} \cong S_n^2(f),$$

if the stochastic process is Gaussian and the spectrum does not change rapidly.

For signals consisting of periodic components, it is also well-known that the *mean* of the Expn (1) spectral estimator of the following signal:

$$x(t) = \sum_{i=1}^{\infty} X_i \cos(2\pi f_i t + \phi_i) \qquad \ldots(2)$$

is as follows:

$$S_p(f) = E\{\hat{S}(f)\} = \sum_{i=1}^{\infty}\frac{X_i^2}{4T_e}\,[W^2(f-f_i) + W^2(f+f_i)], \qquad \ldots(3)$$

where $W(f)$ is the Fourier transform of the window function.

Since periodic signals are generally considered as deterministic ones (Bendat and Piersol, 1971), the *variance* is usually assumed to be zero, and this is often really true. However, as the phase is generally random relating to the observation, $\hat{S}(f)$ has a slightly stochastic nature. The aim of this paper is to develop expressions for the variance caused by the random phase. Four cases will be investigated:

(1) Sine wave with random phase.
(2) Signal with independent, random phase periodic components.
(3) Randomly timed periodic signal.
(4) Random phase periodic signal with additive Gaussian noise.

2 The model

Consider the following random process:

$$x(t) = \sum_{i=1}^{\infty} X_i \cos(2\pi f_i t + \phi_i), \qquad \ldots(4)$$

where ϕ_i has uniform distribution between $(0, 2\pi)$. For the moment ϕ_i and ϕ_j may be interdependent. The windowed Fourier transform of Expn (4) is:

$$X_w(f) = W(f) * \left[\sum_{i=1}^{\infty}\frac{X_i}{2}[\exp(j\phi_i)\,\delta(f-f_i) + \exp(-j\phi_i)\,\delta(f+f_i)]\right]$$

$$= \sum_{i=1}^{\infty}\frac{X_i}{2}[\exp(j\phi_i)\,W(f-f_i) + \exp(-j\phi_i)\,W(f+f_i)]. \qquad \ldots(5)$$

The spectral estimator is:

$$\hat{S}(f) = \frac{1}{T_e}\overline{X_w(f)}\,X_w(f) = \sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\frac{X_iX_k}{4T_e} \times [\![\exp[-j(\phi_i-\phi_k)]\,W(f-f_i)\,W(f-f_k)$$
$$+ \exp[-j(\phi_i+\phi_k)]\,W(f-f_i)\,W(f+f_k)$$
$$+ \exp[j(\phi_i+\phi_k)]\,W(f+f_i)\,W(f-f_k)$$
$$+ \exp[j(\phi_i-\phi_k)]\,W(f+f_i)\,W(f+f_k)]\!]. \qquad \ldots(6)$$

---

* Based on a paper presented at the Fourth Imeko International Symposium on Measurement Theory and its Application to Practice, at Bressanone, Italy, on 9–12 May, 1984.

Since ϕ_i is random, every term of Expn (6) represents a vector on the complex plane, rotating with the changing of ϕ_i or ϕ_k, thus having zero mean, except if the exponent is identically zero. This happens only in the 1st and 4th terms if $i = k$, that is,

$$E\{\hat{S}(f)\} = \sum_{i=1}^{\infty} \frac{X_i^2}{4T_e} [W^2(f-f_i) + W^2(f+f_i)] \qquad \ldots (7)$$

(see Expn 3).

To determine the variance, we shall consider the terms of the following expression [$\hat{S}(f)$ is real; see Expn (1)]:

$$\text{var}\{\hat{S}(f)\} = E[\![\hat{S}(f) - E\{\hat{S}(f)\}]^2]\!] = E\{\hat{S}^2(f)\} - E^2\{\hat{S}(f)\}, \qquad \ldots (8)$$

that is, a fourfold sum must be formed from Expn (6). We shall discuss the terms of this summation in the above listed four special cases.

3 Sine wave with random phase

$$\hat{S}(f) = \frac{X_1^2}{4T_e} [W^2(f-f_1) + W^2(f+f_1) + \exp(j2\phi_1) W(f-f_1) W(f+f_1) + \exp(-j2\phi_1) W(f-f_1) W(f+f_1)]$$

$$= E\{\hat{S}(f)\} + \frac{X_1^2}{4T_e} (2\cos 2\phi_1) W(f-f_1) W(f+f_1).$$

$$\text{var}\{\hat{S}(f)\} = \left(\frac{X_1^2}{4T_e}\right)^2 2 W^2(f-f_1) W^2(f+f_1). \qquad \ldots (9)$$

The relative variance is:

$$\epsilon_v^2(f) = \frac{2W^2(f-f_1) W^2(f+f_1)}{[W^2(f-f_1) + W^2(f+f_1)]^2} \leqslant 0.5. \qquad \ldots (10)$$

Some commonly used windows are illustrated in Fig 1. Expns (9) and (10) are evaluated for these windows in some denominated cases in Figs 2 to 5. We can observe that if the two windows, $W(f-f_1)$ and $W(f+f_1)$ do not overlap significantly, the variance will be small. So we have to use large dynamics windows [e g, the so-called Flat Top (Cox, 1978; Kollár and Nagy, 1982)] to achieve small sidelobes, and we must be careful with low-frequency sinusoidal components: the bandlimit of the spectral window must be smaller than the frequency of the sine wave.

A useful expression for the relative variance can be derived in the domain of the main lobe ($f \approx f_1$) if

$$f_1 >> \frac{1}{T}:$$

since in this domain $W(f-f_1) >> W(f+f_1)$, the relative variance may be approximated as:

$$\epsilon_v^2(f) = \frac{\text{var}\{\hat{S}(f)\}}{E^2\{\hat{S}(f)\}} \cong 2\frac{W^2(f+f_1)}{W^2(f-f_1)} << 1 \qquad \ldots (11)$$

4 Signal with independent, random phase periodic components

We consider the Expn (2) signal, assuming that for $i \neq j$, ϕ_i and ϕ_j are independent. Expn (8) can be evaluated as in Section 3. Squaring (6) we have 16 different combinations of the phases ϕ_{i_1}, ϕ_{k_1}, ϕ_{i_2}, ϕ_{k_2}. Because of the

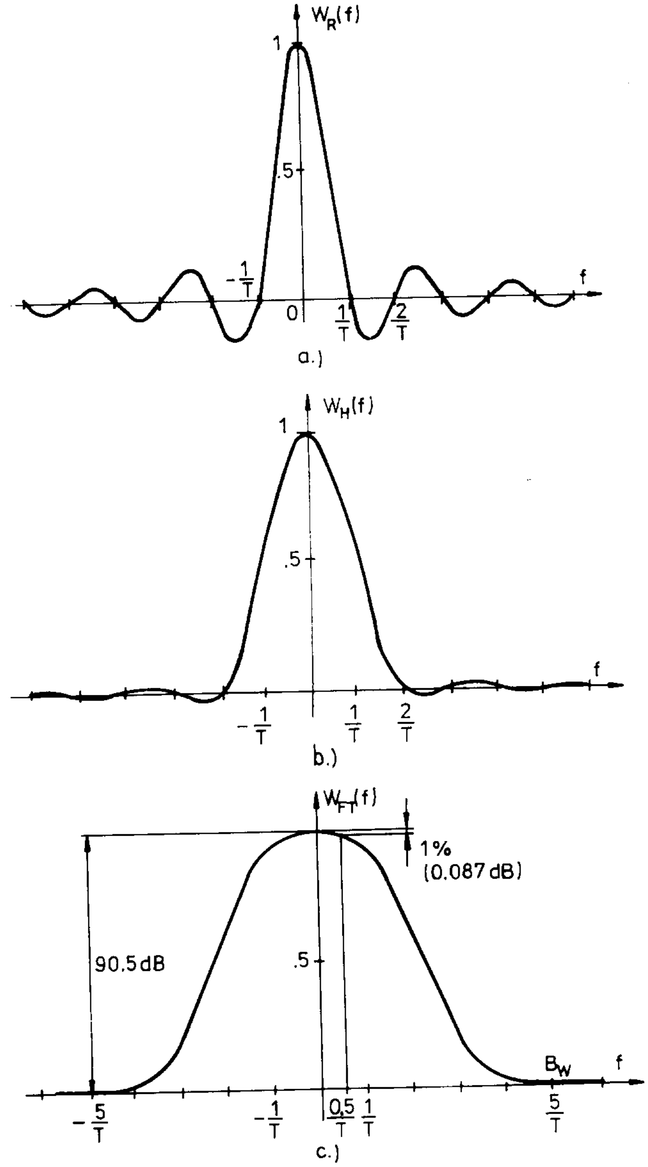

Fig 1 Spectra of windows. (a) Spectrum of the rectangular window. (b) Spectrum of the Hanning window. (c) Spectrum of the Flat Top window.

rotation, the expected value is zero for many terms. It is non-zero only if the exponent is identically zero, that is, the four ϕ-terms by pairs give zero phase.

Terms obtained from the 1st and 4th terms of Expn (6) with the condition $i_1 = k_1$, $i_2 = k_2$ make the $E^2\{\hat{S}(f)\}$ term of Expn (8). So the variance can be calculated by carefully collecting the other zero-exponent terms. The result consists of twofold sums, which can be rewritten into a onefold form:

$$V_p(f) = \text{var}\{\hat{S}(f)\} = \left[\sum_{i=1}^{\infty} \frac{X_i^2}{4T_e} [W^2(f-f_i) + W^2(f+f_i)]\right]^2 + \left[\sum_{i=1}^{\infty} \frac{X_i^2}{4T_e} 2W(f-f_i) W(f+f_i)\right]^2 - \sum_{i=1}^{\infty} \left(\frac{X_i^2}{4T_e}\right)^2 [W^4(f-f_i) + 4W^2(f-f_i) W^2(f+f_i) + W^4(f+f_i)]. \qquad \ldots (12)$$

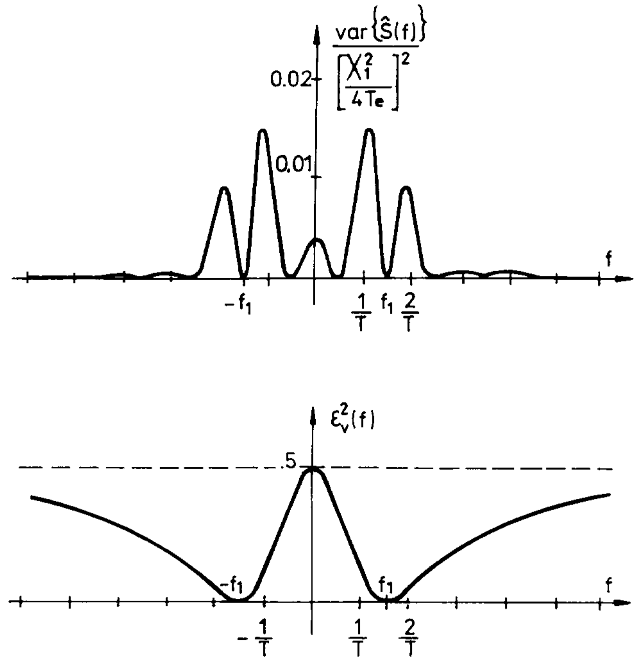

Fig 2 The variance functions in the case of the rectangular window, $f_1 = 1.5/T$

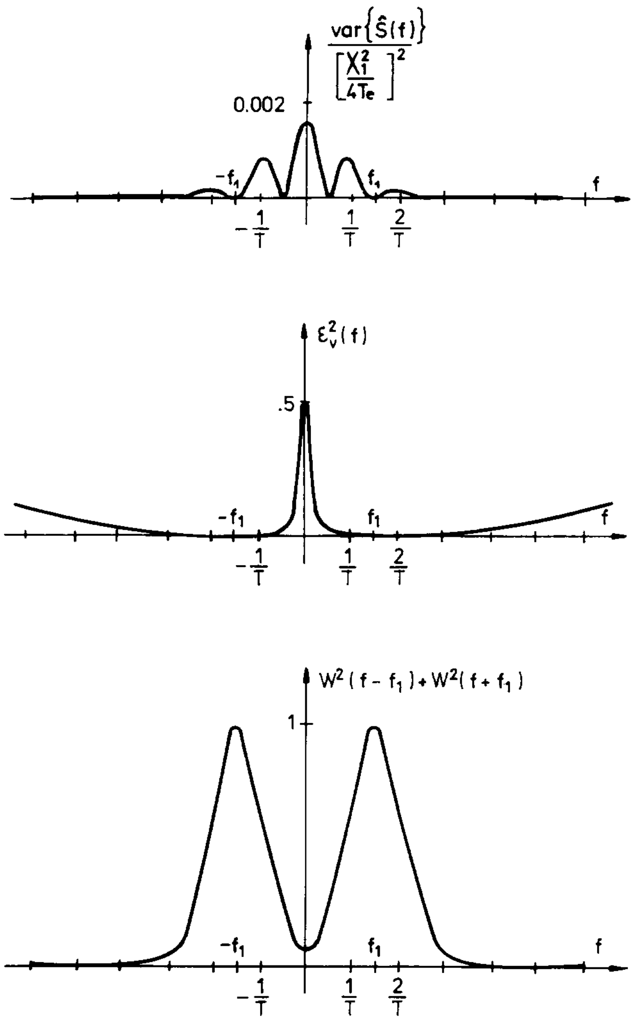

Fig 3 The variance functions and the shape of the spectral estimator in the case of the Hanning window, $f_1 = 1.5/T$

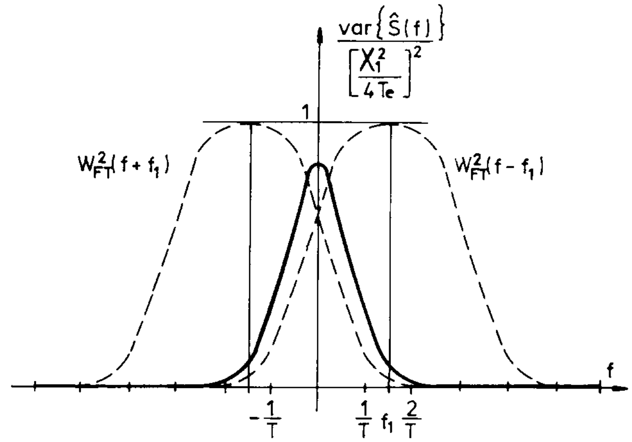

Fig 4 The Expn (9) variance in the case of the Flat Top window, $f_1 = 1.5/T$

Expn (12) may be used to plot variance functions with moderate computing effort. From Expn (12) a simple rule may be read off: every *mixed* product formed from Expn (3) sum plays a role in it. This means that the conclusions of Section 3 concerning small leakage and

$$f_i >> \frac{1}{T}$$

are valid. Still in this case, a widened rule must be applied: two terms of Expn (6) may not be closer to each other than the width of the spectral window, e g, $2B_w$ in the case of the Flat Top one (see Fig 1).

Notice that

$$|f_i - f_k| < 2B_w \qquad \ldots (13)$$

corresponds to the case of frequency beat, and in this light the above result is not any more surprising, although the appearance of the phenomenon also depends on the window shape.

It is an interesting result that the conditon of small variance is similar to the condition of sufficient resolution.

5 Periodic signal with random timing

Random timing means that the periodic signal can be described as follows (disregarding the mean):

$$x(t) = \sum_{i=1}^{\infty} X_i \cos\,[2\pi i f_1(t - t_0) + \Phi_{i0}]$$

$$= \sum_{i=1}^{\infty} X_i \cos\,[i(2\pi f_1 t + \phi) + \Phi_{i0}], \qquad \ldots (14)$$

where t_0 has uniform distribution between

$$\left(0, \frac{1}{f_1}\right)$$

and ϕ between $(0,\ 2\pi)$. In this case the Φ_{i0} initial phases are deterministic parameters. Evaluating Expn (6) we find that instead of the terms

$\exp\,(j\phi_i)$

we obtain terms like

$\exp\,[j(i\phi + \Phi_{i0})]$.

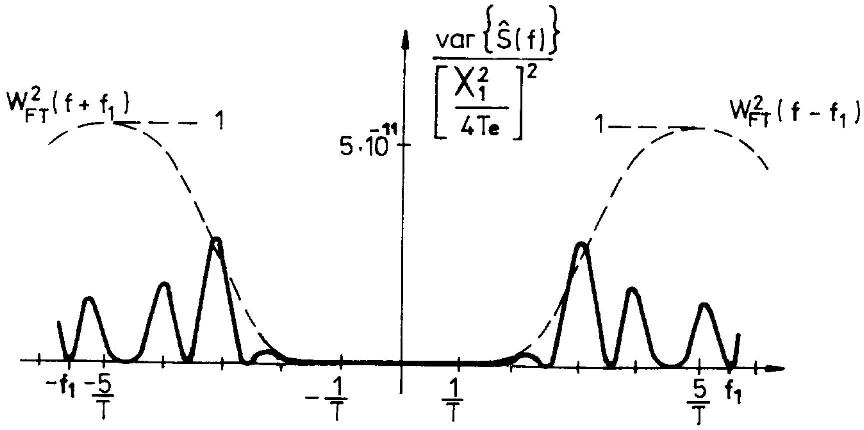

Fig 5 The Expn (9) variance in the case of the Flat Top window, $f_1 = 5.5/T$

It is easy to see that, assuming $\Phi_{i0} = 0$ $[\exp(j\Phi_{i0} = 1)]$, we generally overestimate the variance. So evaluating Expn (8) we may have terms like that:

$$\exp[j(\pm i_1 \pm k_1 \pm i_2 \pm k_2)\phi]. \qquad \ldots(15)$$

These terms, in contrast to Section 4, may be constant even if i_1, k_1, i_2, k_2 are all different, since the exponent may be identically zero.

Therefore it is impossible to express the variance in a simple way like Expn (12). However, in these 'irregular' terms, there are different $W(f)$ terms, and small leakage provides small variance again. So the rules of Section 4 may be applied here, and Expn (12) may give a more or less realistic approximation of the variance, which is expected to be dominated by the neighbouring terms ($|k - i| = 1$) in Expn (6).

6 Additive Gaussian noise

Since the spectral estimator of periodic signals has both deterministic and stochastic components, the resultant variance of the spectral estimator of the mixed signal will not simply be the sum of the two variances, but a more complex expression. Instead of Expn (5), in this case the windowed Fourier transform of the signal will be the following:

$$X_w(f) = \left[\sum_{i=1}^{\infty} \frac{X_i}{2} \exp(j\phi_i)\, W(f - f_i) + \exp(-j\phi_i)\, W(f + f_i)\right] + \xi_1 + j\xi_2, \qquad \ldots(16)$$

where ξ_1 and ξ_2 are independent, Gaussian random variables with zero mean and the following variance:

$$\mathrm{var}\{\xi_1\} \cong \mathrm{var}\{\xi_2\} \cong \frac{T_e}{2} S_n(f).$$

(see Bendat and Piersol, 1971, pp 189–191).

With a similar derivation to that of Sections 2 to 4 (see also Kollár and Nagy, 1982) we obtain:

$$V_m(f) = S_n^2(f) + 2S_n(f)\, S_p(f) + V_p(f). \qquad \ldots(17)$$

In the domain of the main lobes, it is usually the second term of Expn (17) that dominates. This means that, in this domain, the variance may be significantly larger than the generally used $S_n^2(f)$ approximation.

7 References

Bendat, J. S. and Piersol, A. G. 1971. *Random data: Analysis and measurement processes*, John Wiley and Sons Inc, New York.

Cox, R. G. 1978. 'Window functions for spectrum analysis', *Hewlett-Packard Journal,* **29** (13), 10–11.

Kollár, I. and Nagy, F. 1982. 'On the design and use of FFT-based spectrum analysers', *Periodica Polytechnica – Electrical Engineering,* **26** (3–4), 295–315.